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DESCRIPTIVE GEOMETRY

PART I

1

ERRATA.

Fig. 70. (15) The lines o^vq^v and a^ve^v should be full; m^vn^v dotted; and c^vg^v dash and dot.

Fig. 70. (16) The radius of the base of the cone should be ⅝″ instead of ⅞″.

Page 59, last line. For "fourth" read "third."

Page 83, third line from bottom, also last line. For "g^hf^h" read "g^ht^h."

PRINTED BY
GEO. H. ELLIS CO.
BOSTON
1907

DESCRIPTIVE GEOMETRY

PART I

1

BY

CHARLES L. ADAMS

ASSOCIATE PROFESSOR OF DRAWING AND DESCRIPTIVE GEOMETRY
IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY

SECOND EDITION ENLARGED

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PREFACE.

Descriptive geometry as taught in the Massachusetts Institute of Technology is intended to lead to the solution of the graphical problems which ordinarily arise in engineering and architecture; to train the imagination as necessary in these professions; to cultivate the sense of accuracy; and to give facility in drafting and the ability to read working drawings with ease. Consequently this book, an outgrowth of printed notes, treats the subject from the standpoint of the draftsman rather than from that of the mathematician, placing reliance on graphical methods, and omitting analytical methods. Great importance is attached to the drafting exercises which supplement, at the Institute, the use of the book; in these the student works out a set of study plates which give printed data for a large number of original problems and applications.

Considerable work in the projection of solids is given in Chapter VII., since the author believes better results will be obtained if, from time to time, some of this work be combined with the study of the problems on the line and plane. It has also been thought best to give problems in all four quadrants, but the book contains a much larger number of examples in the first and third for the benefit of students in architecture and engineering.

The author desires to express his obligations for general suggestions regarding method and scope to Professor Burton, head of the Department of Drawing in the Institute, who has recently studied abroad the methods of teaching descriptive geometry in the École des Beaux-Arts and other French schools. Mr. Harry C. Bradley, of the Institute, has assisted in the many details of preparation; the selection of data for the examples and the making of the drawings are largely his work.

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DESCRIPTIVE GEOMETRY.

CHAPTER I.

DEFINITION; VALUE OF DESCRIPTIVE GEOMETRY; GENERAL DESCRIPTION OF ITS MODES OF REPRESENTATION.

1. **Descriptive Geometry** is the science of the graphical representation of form, and of the graphical solution of problems relating to form, by the application of geometrical principles.

2. **General Divisions.** The processes of descriptive geometry may be classed in two general divisions, commonly called *Mechanical Perspective* and *Orthographic Projection.*

Mechanical perspective, though it has the more limited application to practical uses, shows more forcibly than orthographic projection the possibilities of descriptive geometry. In perspective drawing the draughtsman, by means of strictly geometrical methods, reproduces the outlines of geometrical solids, or of objects susceptible of geometrical analysis, as these objects would appear when seen by the eye in natural perspective, or when photographed. In orthographic projection, on the other hand, geometrical objects are represented, not as they would appear under natural conditions, but so as to show their actual relations of position, shape, and size.

3. **Value of Descriptive Geometry.** Descriptive geometry is of value for the training it affords and for its practical applications. As training, it develops the power to visualize geometrical relations, and all magnitudes involving three dimensions; that is, to see them in imagination as figures in space — a

power characteristic of successful inventors and designers and of experienced architects and engineers. Descriptive geometry also teaches analytical methods of thought, and assists in the acquirement of a sense of precision.

The practical applications of descriptive geometry are numerous in science and art, but are most frequently met with in engineering, architecture, and the mechanic arts. Here the principles and methods of descriptive geometry are constantly applied: *first*, in the making of drawings used in connection with building construction and shop work; *second*, in the graphical determination of special forms such as occur in woodwork, metal-work, and stone-cutting.

Mechanical perspective has a valuable although somewhat limited commercial application. It is used principally in connection with architectural designs, to show the prospective appearance of buildings and interiors. A knowledge of the principles of mechanical perspective is also of distinct value to the artist and the freehand draughtsman. It supplements observation, and is indispensable as a means of verifying freehand drawings of geometrical objects.

4. **Representation.** In the representation of form, descriptive geometry is concerned with three kinds of drawing; namely, *pictorial*, *pseudo-pictorial*, and *constructive*.

A pictorial drawing is one which represents geometrically the outlines of an object strictly as seen by the eye.

A pseudo-pictorial drawing is a conventional representation, which suggests the natural appearance of an object, but at the same time gives certain facts of its actual form.

A constructive drawing represents an object so as to show its actual form and measurements.

In order to make, geometrically, a pictorial drawing, it is necessary to understand constructive drawing, but, in describing these two kinds of drawing, it is convenient to consider them in the order stated above.

5. **Pictorial Drawing.** The theory of pictorial representation depends in part on certain laws of vision. Hence

a few facts taken from Optics may here be introduced to advantage.

Let it be understood that the word "object," as here used, always means the boundaries of a geometrical solid or of an object composed of geometrical solids.

THE EYE; CONE OF RAYS; RETINAL IMAGE. In order to eliminate the complications of binocular vision, it will be supposed that we always observe with but one eye. The rays of light transmitted to the eye, Fig. 1, may be represented by straight lines. They pass through the circular aperture (pupil), and, converging to the point A, form the *cone of rays* FAG. Passing through the point A, the rays diverge in the form of a second cone, and fall upon the back of the eye (retina), where they form what are called ***retinal images***. The common axis of these two cones is termed the ***principal visual ray*** (PVR).

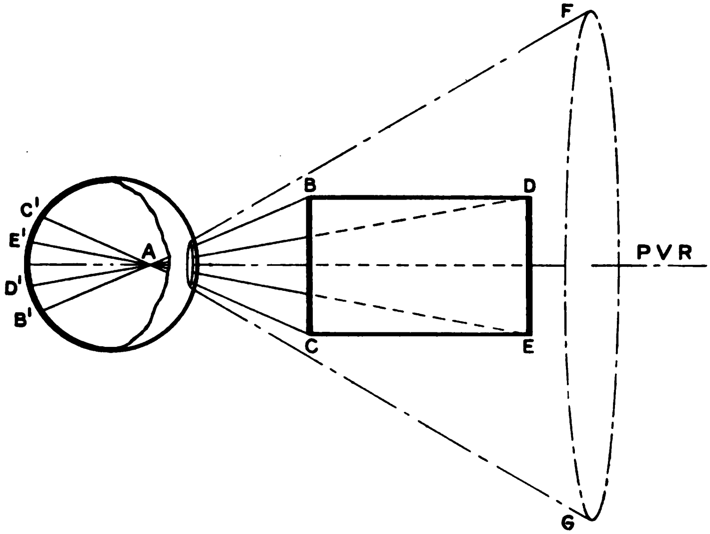


Fig. 1.

If an object be placed before the eye, then, geometrically speaking, the retinal image of this object is the figure formed by the intersection of the rays of light from the object, and the (spherical) surface of the retina.

Natural Perspective. If now two equal vertical lines, as *BC* and *DE*, Fig. 1, are held before the eye, and in such positions that one line will not hide the other, then the line *BC* will appear to the eye longer than the line *DE*, since the retinal image, *B'C'*, of line *BC* is longer than the retinal image *D'E'* of line *DE*. Consider the lines *BC* and *DE* to be connected, as shown, by the horizontals *BD* and *CE*. Although actually parallel, the lines *BD* and *CE* will *appear* to converge, since they connect *BC* and *DE*, two lines of *apparently* unequal length. Therefore, the figure *BCED*, in reality a rectangle, will appear as a trapezoid. This apparent change of figure is due to *natural perspective*, and illustrates the kind of changes which occur in the appearance of lines and surfaces, resulting from the convergence of the visual rays.

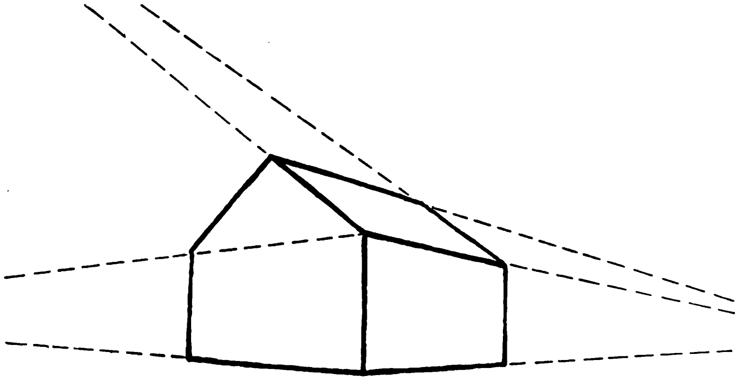


Fig. 2.

The effect of natural perspective may be illustrated further by the freehand sketch of the appearance of the outlines of a house (Fig. 2). Note the apparent difference in length of the vertical edges, which in the building are equal. Note, also, the apparent convergence of lines actually parallel, and the changed figure of surfaces actually rectangular.

REPRESENTATION OF NATURAL PERSPECTIVE BY PROJECTION. *A perspective drawing* is one which reproduces the natural perspective of an object, as seen from any *fixed* position. This position is variously called the "position of the spectator," the "point of view," or the "station-point."

Experiment. Let it be imagined that we are looking through a window at the actual house sketched for Fig. 2, and that the eye is placed in the same position relative to the house as that from which the sketch was made. With a wax crayon, and while looking at a corner of the house, we may place on the glass a point which covers or hides from us the corner in question. The point on the glass lies in the visual ray transmitted from the corner of the house to the eye. The point on the glass, however, being opaque, intercepts the visual ray from the point in the object. Hence the retinal image of the point in the object is now replaced by the retinal image of the point on the glass. Proceeding in like manner, without changing the position of the eye, we may intercept rays from other points in the outlines of the house, connect them by straight lines, and thus obtain on the glass a figure which produces a retinal image having the same outlines as the retinal image caused by the object itself. The drawing on the glass is obviously a *perspective drawing,* since it reproduces the apparent relations of the lines and surfaces of the house, as seen from a fixed position.

In the language of descriptive geometry the drawing on the glass would be called a *projection* of the house; the glass, a *plane of projection;* the rays passing from the house to the eye, *line projectors;* and, since the projectors converge to a point (the eye being so regarded), the drawing on the glass would be called a *convergent* projection.

6. Systems of Projection. (*a*) *Convergent Projection.* The nature of a geometrical convergent projection is shown in Fig. 3. The line *CD* represents a plane of projection seen edgewise and from its right-hand side. The object projected, *B*, is a cube placed, on a horizontal surface, so that its vertical faces make equal angles with the plane of projection. The eye is represented by a point, *E*, which is here taken above the cube and opposite its centre. If the points where the line projectors, 1, 2, . . . 5, pierce the plane are found and connected by straight lines, then the result—a *convergent projection* or *mechanical perspective*—is a drawing similar to that shown at A_1.

CONVERGENT PROJECTION

Side View of Plane of Projection, Projectors, and Cube at B

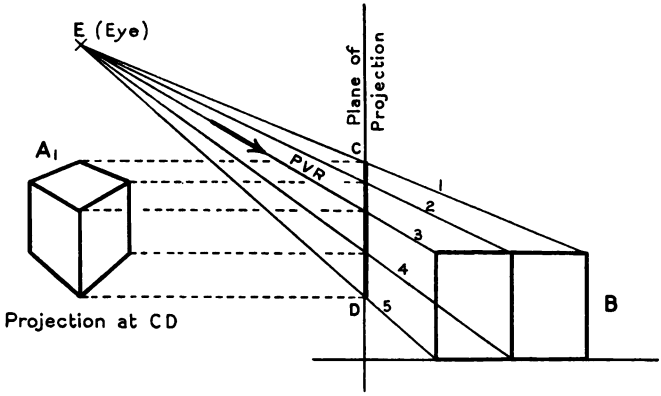


Fig. 3.

(*b*) *Parallel Projection.* It will be seen in Fig. 3 that the convergence of the visual rays distorts the actual figure and measurements of the object. In order to avoid the distortion resulting from this convergence, descriptive geometry assumes that objects can be seen with parallel rays, as illustrated in Fig. 4.

Let the line 3, Fig. 3, represent the principal visual ray of the eye at *E;* and let the same visual ray be represented in Fig. 4 by the line 3. If now, in viewing the cube, Fig. 4, it is imagined that the eye, instead of remaining fixed, as in Fig. 3, is moved so that the line of sight or principal visual ray will always be parallel to the ray 3, then it may be said that the cube is seen with *parallel rays*. If, in Fig. 4, the points where the line projectors pierce the plane of projection are found and connected, then the result—a *parallel projection*—is a drawing similar to that shown at A_2. Compare A_1 and A_2—representations of the same object in the same position relative to the plane of projection—and note the difference in the size and shape of the two projections. Note, also, that the convergent lines in A_1 are replaced by parallel lines in A_2.

PARALLEL PROJECTION

Side View of Plane of Projection, Projectors, and Cube at B

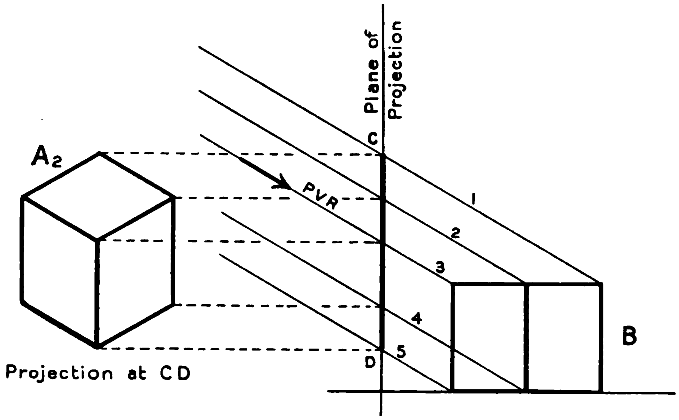
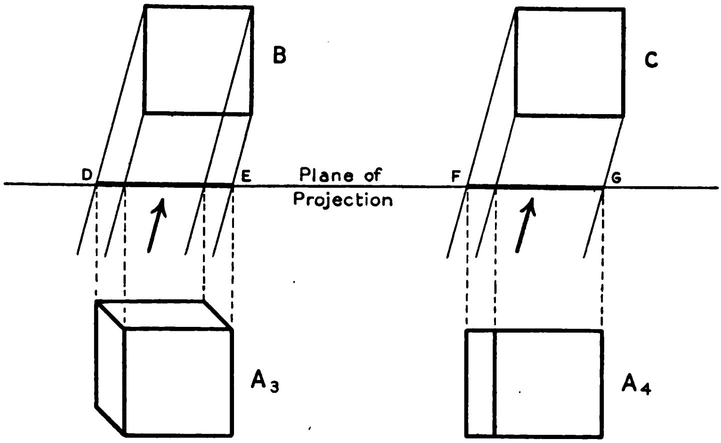


Fig. 4.

(*c*) *Systems of Parallel Projection.* Parallel projection is divided into two general systems: (1) *oblique* projection, in which the projectors are *inclined* to the plane of projection; and (2) *orthogonal* projection, in which the projectors are *perpendicular* to the plane of projection.

The drawings A_2, A_3, and A_4 are examples of oblique projection. The drawings A_5 and A_6 are examples of orthogonal projection.



PARALLEL PROJECTION
(OBLIQUE)

Top View of Plane of Projection, Projectors, and Cube at B and C

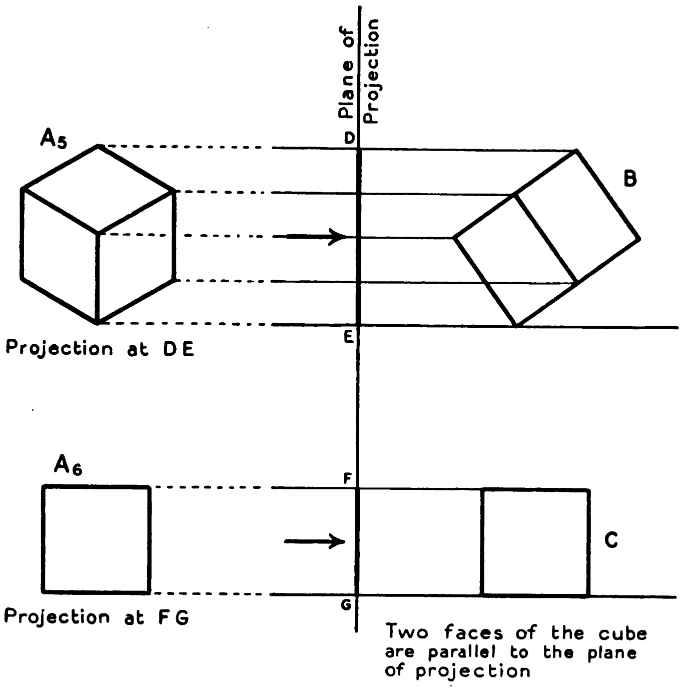
Projection at DE. The projectors are inclined to the front, side, and top faces of the cube

Projection at FG. The projectors are inclined to the front and side, but parallel to the top face of the cube

Fig. 5.

7. Pseudo-pictorial Drawing. It will be observed that the drawings A_2, A_3, and A_5 suggest both the natural appearance and the actual form of the cube. With these and similar projections as a basis, various conventional systems of representation have been derived, and these are classed under the general head pseudo-pictorial representation. The practical advantage of this mode of representation lies in the fact that its methods are easier and quicker than those of perspective.

8. Constructive Drawing. It should be noted that, up to this point, objects have been projected upon but one plane,



PARALLEL PROJECTION
(ORTHOGONAL)

Side View of Plane of Projection, Projectors, and Cube at B and C

Two faces of the cube are parallel to the plane of projection

Fig. 6.

giving a single projection, or *view*, of the object. When, however, it is required to represent *completely* the actual form and measurements of an object, two or more views of it are necessary. For example, a complete representation of the actual form of the object given in Fig. 7 requires all the views shown in Fig. 8. These views might be orthogonal projections on one plane, the object being successively placed in the position necessary to obtain each view. In descriptive geometry, however, the object would be regarded as fixed, and its several views would be orthogonal projections on two or more non-parallel planes. The representation of actual form thus effected is called *constructive drawing*, and it is with the fundamental principles and methods of this subject that the study of descriptive geometry begins.

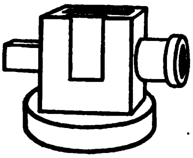


Fig. 7.

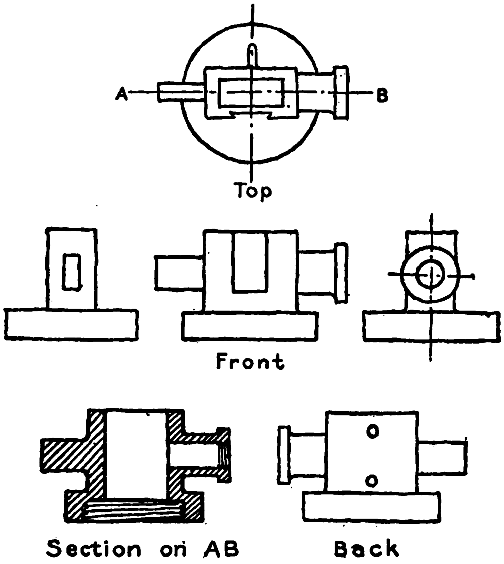


Fig. 8.

ORTHOGONAL PROJECTION.

CHAPTER II.

TERMS; THE CO-ORDINATE PLANES; PROJECTIONS ON THE CO-ORDINATE PLANES.

9. Terms. (*a*) If a straight line be passed through any point in space, perpendicular to a plane, then the point in space is said to be *orthogonally projected* upon the plane at the point where the perpendicular intersects the plane (Fig. 9). The perpendicular aa^r is called the *projector* (*line projector*) of the point *a;* the plane *R* is called the *plane of projection;* and the point a^r the *projection* * of the point *a*.

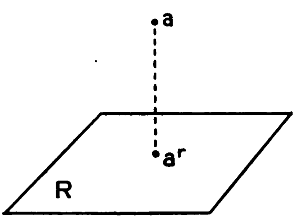


Fig. 9.

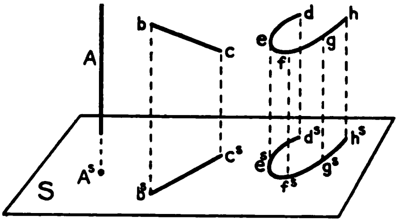


Fig. 10.

(*b*) The projection of any line is a line composed of the projections of all the points of the given line (Fig. 10).

The projection of any plane figure is the figure formed by the projections of the boundary lines of the figure projected, Fig. 11.

The projection of any solid is composed of the projections of all its points, lines, or surfaces.

---

* "Projection," used without a qualifying adjective, always means an *orthogonal projection*.

(*c*) A straight line may be projected by a plane; a curved line, by a curved surface; and a plane figure, by a solid.

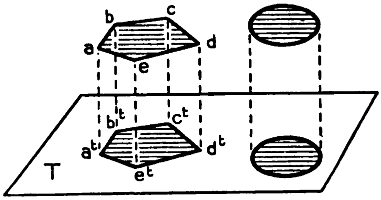


Fig. 11.

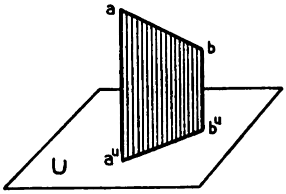


Fig. 12.

A *plane projector* is the plane composed of all the line projectors of a straight line. The plane abb^ua^u, Fig. 12, is the plane projector of the line *ab*.

A *solid projector* is the solid included by the surface or surfaces, plane or curved, which project the boundary lines of a plane figure. The solid projectors of the plane figures *S* and *T*, Fig. 13, are the prism and cylinder respectively.

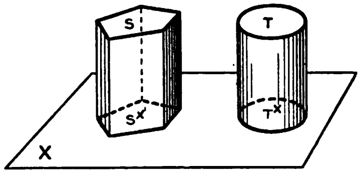


Fig. 13.

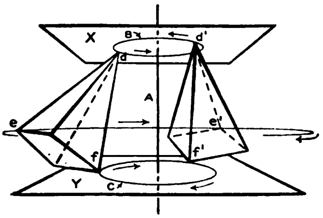


Fig. 14.

(*d*) *Axis of revolution; plane of revolution.* If a line, plane, or solid, be revolved about any fixed straight line as an axis, then each point moves in a circle lying in a plane perpendicular to the axis, and having its centre in the axis. The axis is called the *axis of revolution*, and the perpendicular planes are called *planes of revolution.* For example, in Fig. 14, the line *A* is the axis of revolution, the circles *B* and *C* are the paths of the points *d* and *f* respectively, and the planes *X* and *Y* are planes of revolution.

(*e*) A *trace* is an intersection with a plane of projection. The trace of a line is a point; the trace of a surface is a line; and the trace of a solid is a plane figure. (See Figs. 15, 16, and 17.)

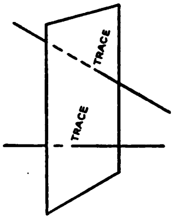


Fig. 15.

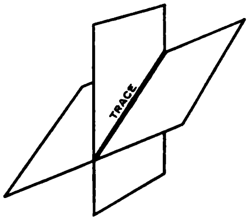


Fig. 16.

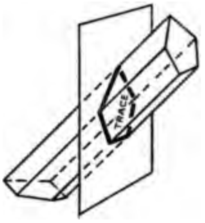


Fig. 17.

10. The Co-ordinate Planes. These are three imaginary, mutually perpendicular planes, conceived in space. They are the principal planes of projection used in the representation of objects, and the solution of problems. To see in space the co-ordinate planes, together with any geometrical figures or relations to be projected on the planes, requires a certain definite power of imagination. This sort of imagination is also necessary in making a projection drawing, since in this drawing the mutually perpendicular co-ordinate planes and all projections thereon must be conceived to lie in *one* plane, as the surface of the drawing-paper or blackboard. Further, in order to visualize or *read* a projection drawing, it is necessary, by a converse process of imagination, to see the object or geometrical relations represented, as if actually *existing in space.*

Let the student *see in imagination* three intersecting planes in space,—the first horizontal, the second vertical, and the third perpendicular to the first and second. These, the *co-ordinate planes*, Fig. 18, are severally distinguished as the *horizontal plane* (*H*), the *vertical plane* (*V*), and the *profile plane* (*P*). The four dihedral angles formed by *H* and *V* are termed the *first, second, third,* and *fourth quadrants* or *angles,* respectively. (See Fig. 18.) The intersection of *H* and *V* is called *the ground line* (*GL*).

Next, let the student imagine the co-ordinate planes in space as they would appear if seen with parallel rays and in the general directions indicated by the arrows (Fig. 18).

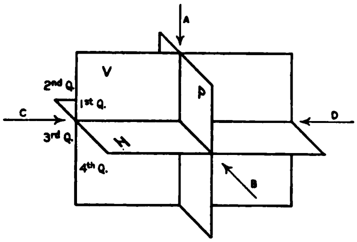


Fig. 18.

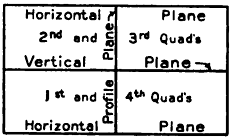


Fig. 19.

A

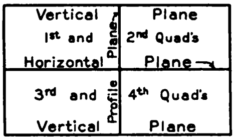


Fig. 20.

B

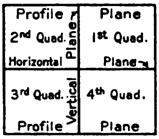


Fig. 21.

C

(*a*) The arrow *A* indicates that the observer is supposed to look *downward*, toward, and in a direction perpendicular to *H*. Hence: (1) *H* is *below* the observer, the first and second quadrants are *above H*, and the third and fourth quadrants are *below H*. (2) *H* is seen flatwise; *V* is seen edgewise, as a line covering or hiding *GL;* and *P* is seen edgewise, as a line perpendicular to *GL*.

(*b*) The arrow *B* indicates that the observer is supposed to look *toward* and in a direction perpendicular to *V*. Hence it follows: (1) *V* is *in front* of the observer; the first and fourth quadrants are *in front* of *V;* and the second and third quadrants are *behind V*. (2) *V* is seen flatwise; *H* is seen edgewise, that is, as a *line*, covering the ground line; and *P* is seen edgewise, as a line perpendicular to *GL*.

(*c*) The arrow *C* indicates that the observer is supposed to look from *left to right*, and in a direction perpendicular to *P*. Hence: (1) *P* is in front of the observer; the first and fourth quadrants are situated to the ***right*** of the observer; and the second and third quadrants are to his *left.* (2) P is seen flatwise; H and V are seen edgewise, as two lines at right angles; and the ground line is seen endwise,—that is, as a point.

(*d*) The arrow D indicates that the observer is supposed to look from ***right to left,*** and in a direction perpendicular to P. Hence, from this point of view, the first and fourth quadrants are to his left, while the second and third quadrants are to his right.

11. Representation of the Co-ordinate Planes. Having thus conceived the co-ordinate planes in space and the directions in which they are supposed to be viewed, the next step is to learn how the planes are expressed by a single surface.

In making a projection drawing, the surface of the drawing-paper or blackboard represents all three of the co-ordinate planes. For present purposes the drawing-paper and blackboard may be expressed by the page of this book. In conceiving the page to represent the co-ordinate planes, either of two methods may be adopted: *first,* the page may represent the planes as seen by changing the direction of sight relative to the planes, which are regarded as *fixed;* or, *second,* the eye may be supposed to look in one direction, and the different views of the planes are obtained by ***revolving the planes.*** In practice, these two methods are often combined.

First Method. (*a*) Let the portion of the page included by Fig. 19 represent H. Hold Fig. 19 horizontal, and imagine two cards held perpendicular to the page, and passing through it in the lines lettered "vertical plane" and "profile plane," respectively. Looking downward and in a direction perpendicular to the page, H (the page) is seen flatwise, V (one card) is seen edgewise, and P (the second card) is also seen edgewise.

Let the portion of the page included by Fig. 20 represent V. In this figure it is assumed that the observer looks in a horizontal direction. Hold Fig. 20 vertical, and imagine two cards, each perpendicular to the page, and passing through it in the lines lettered "horizontal plane" and "profile plane," respectively. Looking in a direction perpendicular to the page, *V* (the page) is seen flatwise, *H* (the first card) is seen edgewise, and *P* (the second card) is also seen edgewise.

The portion of the page included by Fig. 21 represents *P*. The eye looks in a horizontal direction, but perpendicular to the direction in which Fig. 20 is viewed, and in this case from left to right. Holding the page vertical, *P* (the page) is seen flatwise, and *H* and *V* are each seen edgewise.

To summarize the preceding, the co-ordinate planes—represented by the successive positions in which the page and the cards were held—are fixed. There are three directions of sight, and one surface (the page of the book) contains the three different views of the planes of projection.

Second Method. (*b*) Hold the page of the book horizontal, and imagine two cards, each perpendicular to the page, passing through the respective lines *GL* and *HP*, Fig. 23. Looking in a direction perpendicular to *H* (the portion of the page included by Fig. 23), *H* is seen flatwise, and *V* and *P* edgewise. Now imagine *V* (the first card) to be revolved about *GL* until it coincides with *H*, as indicated by the arrows *A*

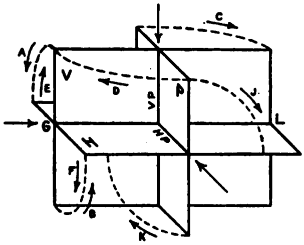


Fig. 22.

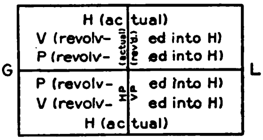


Fig. 23.

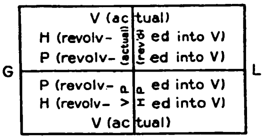


Fig. 24.

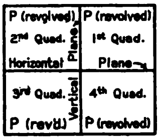


Fig. 25.

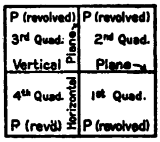


Fig. 26.

and *B*, Fig. 22. The page of the book now represents not only *H*, but *V* revolved into *H* as well (indicated by the terms "*H* actual" and "*V* revolved"). Also imagine the second card to be revolved about *HP* into *H*, as indicated by the arrows *J* and *K*, Fig. 22. The page now represents not only *H* and *V* revolved into *H*, but also *P* revolved into *H* ("*P* revolved").

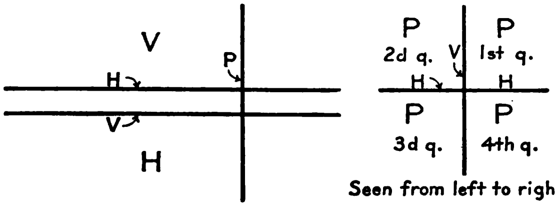
To summarize the preceding, there is but one direction of sight in Fig. 23. The page of the book represents all three co-ordinate planes; that is, *H* actual, *V* revolved, and *P* revolved.

(*c*) In Fig. 23 it is assumed that the book is held so as to bring the page horizontal—the usual position of the drawing-paper. But in the case of a drawing made on a vertical surface, as the blackboard, the page held vertical would naturally be taken to express *V*. Hence, if the page, Fig. 24, be held so as to represent *V*, then *H* would be revolved about *GL* into *V*, as indicated by the arrows *E* and *F* (Fig. 22); and *P* would be revolved about *VP* into *V*, as indicated by the arrows *C* and *D*.

(*d*) In Fig. 25 the co-ordinate planes are represented as having been revolved through 90°, about a *vertical axis*, *VP*, Fig. 22. As the result of this revolution, the page, held vertical and viewed flatwise, represents *P*. The planes *H* and *V*, therefore, are seen edgewise.

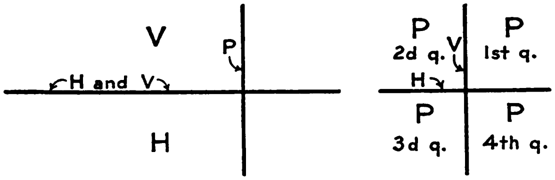
(*e*) In Fig. 26 the co-ordinate planes are represented as having been revolved about a *horizontal axis*, *HP*, Fig. 22, to bring *P* into the plane of the page held horizontal. The difference in the positions of the quadrants, Figs. 25 and 26, should be especially noted.

(*f*) It should be observed in Figs. 23 and 24 that, whether *V* is revolved into *H* or *vice versa*, the page above *GL* represents the *upper* portion of *V*, and the *back* portion of *H*; while the page below *GL* represents the *lower* portion of *V* and the *front* portion of *H*. It should also be noted that, if viewed according to the *first* method, *GL*, Fig. 23, represents *V* seen edgewise.



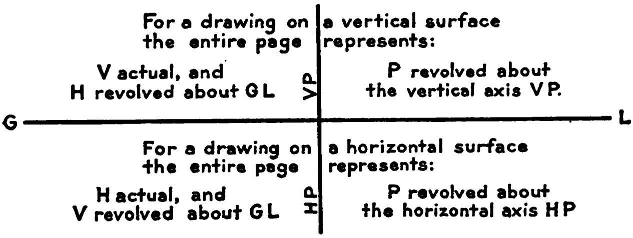
Seen from left to right.

Fig. 27.



Seen from left to right.

Fig. 28.



For a drawing on the entire page a vertical surface represents:

V actual, and H revolved about G L

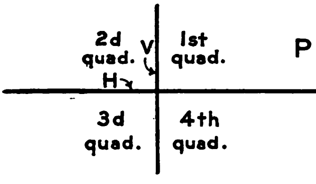
P revolved about the vertical axis V P.

For a drawing on the entire page a horizontal surface represents:

H actual, and V revolved about G L

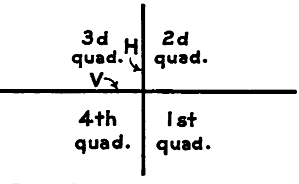
P revolved about the horizontal axis H P

Fig. 29.



P revolved about a vertical axis into the page regarded as vertical. The portion of P in front of V is here revolved to the right.

Fig. 30.



P revolved about a horizontal axis into the page regarded as horizontal. The portion of P above H is here revolved to the right.

Fig. 31.

Hence the part of Fig. 23 *above GL* represents space *behind V*,—that is, the second and third quadrants; while the part below *GL* represents space *in front* of *V*, or the first and fourth quadrants. Similarly, the part of Fig. 24 *above GL* represents the first and second quadrants, while the part *below GL* represents the third and fourth quadrants.

Exercise 1. Before proceeding, the student should practise till he can easily visualize by the first method Figs. 19, 20, and 21, independently of the text, and Fig. 18. He should next visualize, by the second method, the revolutions necessary to give Figs. 23, 24, 25, and 26, and then the counter-revolutions as necessary to see in imagination the planes in space, and the four quadrants (Fig. 22).

12. Working Representations of the Co-ordinate Planes. As an aid in describing them, the co-ordinate planes have thus far been regarded as having boundaries, Figs. 18–26. The planes must now be regarded as indefinite in extent, and be expressed in the general forms shown in Figs. 27–31.

Each of the forms, Figs. 27 and 28, represents Figs. 19, 20, and 21 combined, the only difference in the two forms being that in the first form, Fig. 27—called the *double ground line*—the edge views of *H* and *V* are represented by separate lines, while in the second form, Fig. 28, a single line represents the edge view of both *H* and *V*. These forms should be read according to the *first method*, which, it will be remembered, assumes that the planes are viewed by changing the position of the eye relative to the planes.

Exercise 2. Holding the page horizontal to express *H*, see the horizontal line *V*, Fig. 27, as the edge view of *V*. Hold the page vertical, to represent *V*, and see the line *H* as the edge view of *H*. Proceed in like manner to visualize all the planes as indicated by the letters, Figs. 27 and 28.

In Figs. 29, 30, and 31 are combined the views Figs. 23–26, illustrating the *second method* (Art. 11).

Exercise 3. Holding the page to represent horizontal and vertical drawing surfaces, visualize the revolutions and counter-revolutions indicated by the lettering on the cut, Fig. 29. Visualize the conditions which give the different positions of the quadrants, Figs. 30 and 31.

13. Association of Imaginary with Actual Conditions. It will be observed that the co-ordinate planes correspond to the two directions most often used in all engineering and building operations; namely, the *horizontal* and the *vertical.* Hence it frequently occurs in practical drawing that one or more of the co-ordinate planes have their counterpart in actual surfaces. Thus, for example, in the contour map of the hill, Fig. 32, the horizontal plane is represented by

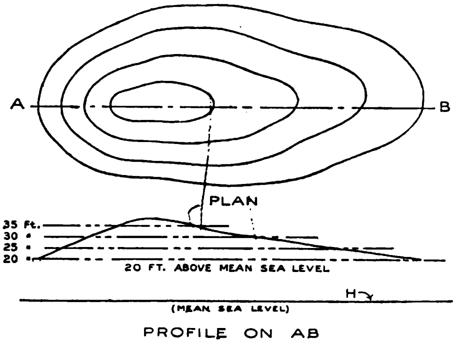


Fig. 32.

mean sea-level at a particular locality, the lines of equal height —as of 20, 25, 30, and 35 feet—being measured from *H*, or the sea-level. Another example is given in Fig. 33, where *H*, *V*, and *P* correspond to the floor and two walls of a room. In Fig. 34 the floor and side wall are represented as revolved into the plane of the back wall, and show the objects in the room, supposed to be projected on the floor and walls.

On beginning study, the student may find it necessary to represent the co-ordinate planes and objects projected on them by an improvised model. But reference to a model, and even to the pictorial diagrams here given, should soon be discontinued, as the practice only tends to defeat one of the principal objects of the study of descriptive geometry,—the *development of the power to visualize.* Far better than the use of a model is the habit of associating the co-ordinate planes and given data with one's immediate surroundings.

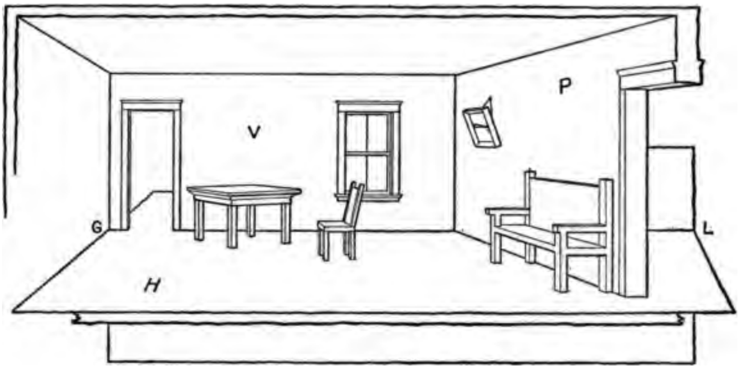


Fig. 33.

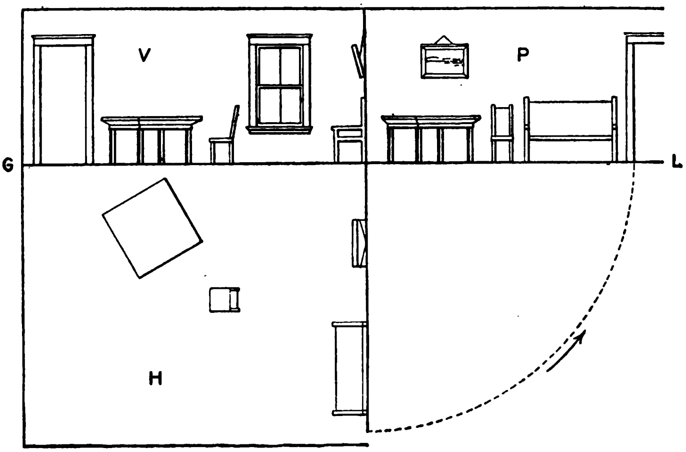


Fig. 34.

14. Projections on the Co-ordinate Planes. *Projections of a point in space.* In Fig. 35 is shown a point, *a*, projected on *H*, *V*, and *P*. The points a^h, a^v, and a^p are called the *horizontal*, *vertical*, and *profile projections* of the point *a*. In Fig. 36 the page represents *V*, Fig. 35, together with *H*, *P*, and the projections thereon revolved into *V*.

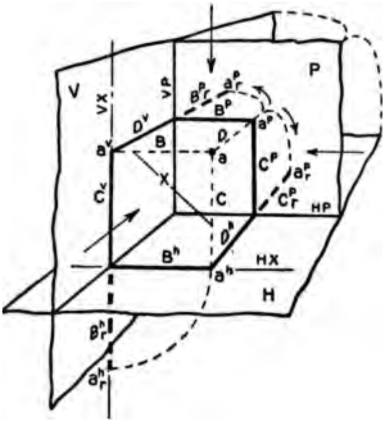


Fig. 35.

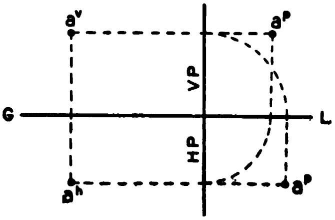


Fig. 36.

It will be seen that two revolved positions of *P* are indicated; *first*, *P* is revolved into *V* (as shown in Fig. 35), carrying a^p to the position above *GL*, Fig. 36; *second*, *P* is revolved first into *H* (as shown in Fig. 35), and then, with *H*, it is revolved into *V*, carrying a^p to the position below *GL*, Fig. 36.

15. Ruled Projectors. Certain relations between the projections on the co-ordinate planes of a point in space can be determined *in the drawing* by means of straight lines, such as $a^v a^p$, $a^h a^p$, and $a^v a^h$, Fig. 36. For example, the student can easily prove, with the aid of Fig. 35, that the line connecting the projections a^v and a^p, Fig. 36, since it is composed of D^v and B^p, projections of the space projectors *D* and *B*, must be a straight line, and parallel to *GL*. Likewise the line $a^h a^p$ is a straight line parallel to *GL*, and the line $a^v a^h$ is a straight line perpendicular to *GL*. As these straight lines may be used to project from one point to another in the drawing, they are called *ruled projectors*—as distinguished from space projectors.

16. The Revolution of a Point about a Line, as shown by Projections. Let a point, *a*, Fig. 37, revolve about an axis *B*, perpendicular to *H*. Since the axis is perpendicular to *H*, its H-projection will be a point, B^h, and its V-projection a straight line, B^v, perpendicular to *GL*. Assume that the given point, *a*, is connected with the axis *B* by a straight line, *ac*, perpendicular to the axis. Revolving *ac* about the axis, the point *a* moves in a circle, *D*, and the line *ac* produced generates the plane of revolution, *X*, perpendicular to both *B* and *V*. The projection of any plane perpendicular to a co-ordinate plane is a straight line, and in the present case, since plane *X* is perpendicular to *B*, its projection *VX* must be perpendicular to B^v. But plane *X* contains the circle *D*, hence D^v must be perpendicular to B^v. Furthermore, since plane *X* containing the circle *D* is parallel to *H*, D^h must be equal to the circle *D*, described from c^h as centre. Finally, revolving *H* about *GL* into *V*, the centre c^h falls at ${c^h}_r$ vertically below c^v (see also Fig. 38). In Fig.

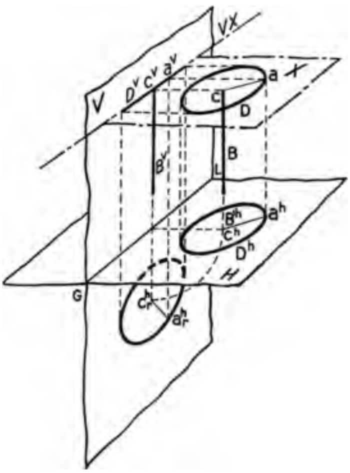


Fig. 37.

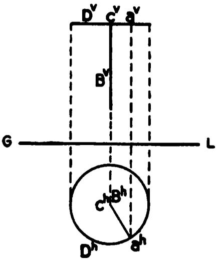


Fig. 38.

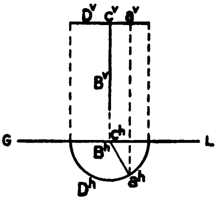


Fig. 39.

39, B^h lies in GL, consequently the axis of revolution lies in V.

Exercise 4. From inspection of Fig. 38 see in space all conditions represented in Fig. 37.

17. **Notation.** A point in space will be indicated by a small (lower case) letter; a right line in space by a capital chosen from A to N. A projection of a point or a line is indicated by the letter used for the point or line in space, combined with an index letter indicating the plane of projection, thus:—

Point in space, a; its projection on V is written a^v.

Line in space, A; its projection on P is written A^p.

The fact that a point, a line, or a projection of either, has been revolved into a co-ordinate plane, is indicated by the subscript r; thus a, a^p, and A^p revolved into V are written a_r, $a^p{}_r$, and $A^p{}_r$.

A plane in space will be indicated by a capital letter chosen from N to Z. A trace of a plane (Art. 9, *e*) takes the same letter as the plane in space, preceded by the letter indicating the co-ordinate plane containing the trace; thus the horizontal trace (intersection with H) of a plane Q is written HQ.

18. On pages 25–28 are given pictorial diagrams showing a prism, A, situated in each of the four quadrants, the projections of the prism on the co-ordinate planes, and certain projections of the prism revolved into either H or V. Observe that, in the several revolutions of P into H or V, the plane P, together with the direction of sight for P, are always turned in the direction which will cause the direction of sight to correspond with that of the plane into which P is revolved. The student should compare the working projections with the accompanying diagram, and note particularly the statements printed below the figures.

Exercise 5. (*a*) Reading Fig. 41, according to the *second method* and preferably without looking at Fig. 40, see in space the *co-ordinate planes*, the *prism*, and *its projections on the planes*.

(*b*) The same applied to Figs. 43, 45, and 47.

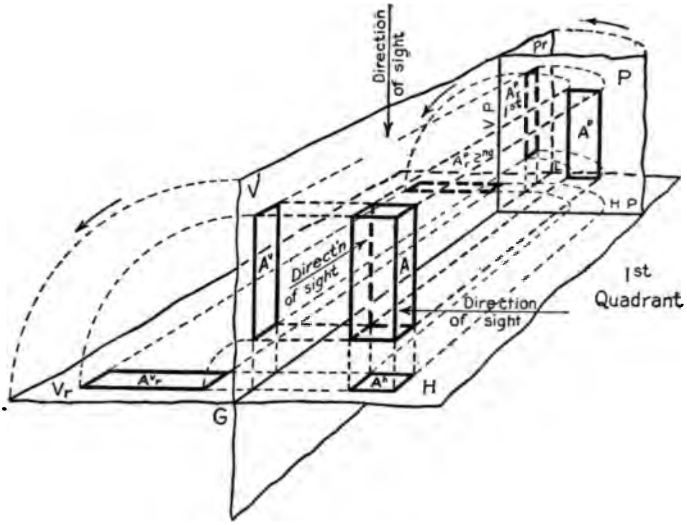


Fig. 40.

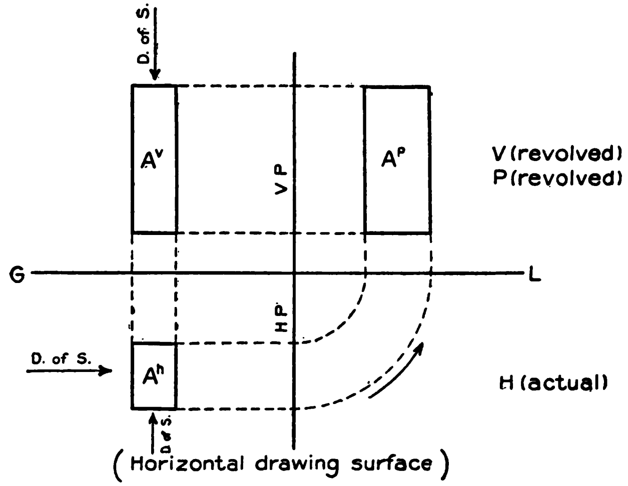


Fig. 41.

The object is between the eye and each co-ordinate plane.
V is revolved into *H*.
P is revolved, first into *V*, and then into *H*.

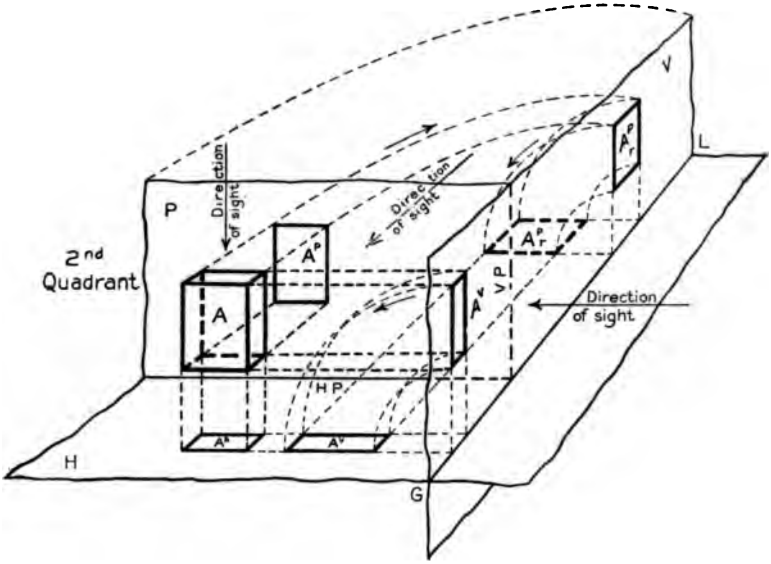


Fig. 42.

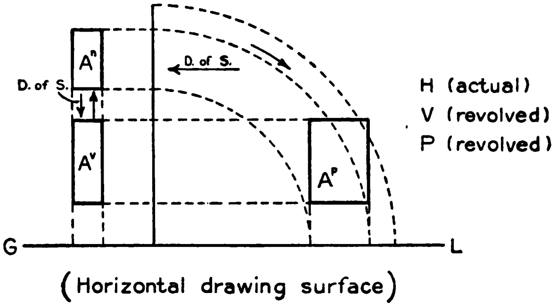


Fig. 43.

V is between the eye and the object.

P is between the eye and the object.

The object is between the eye and *H*.

V is revolved into *H*.

P is revolved, first into *V*, and then into *H*.

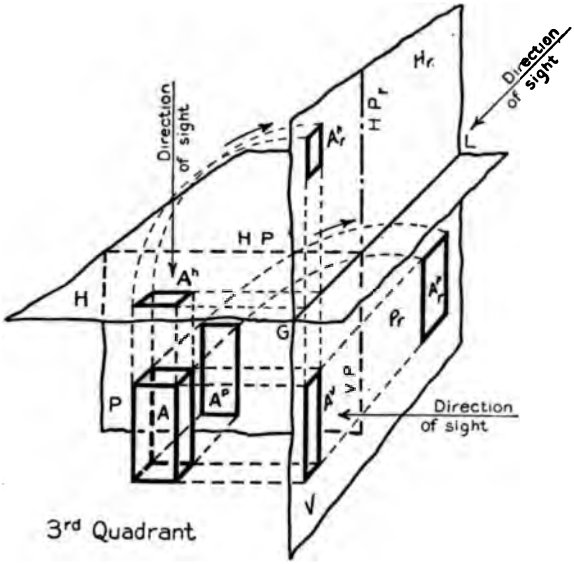


Fig. 44.

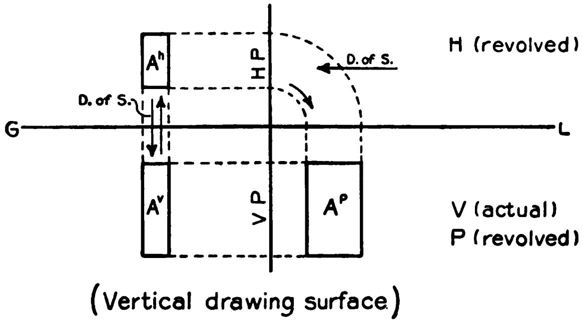


Fig. 45.

Each co-ordinate plane is between the eye and the object.
H is revolved into *V*.
P is revolved into *V*.

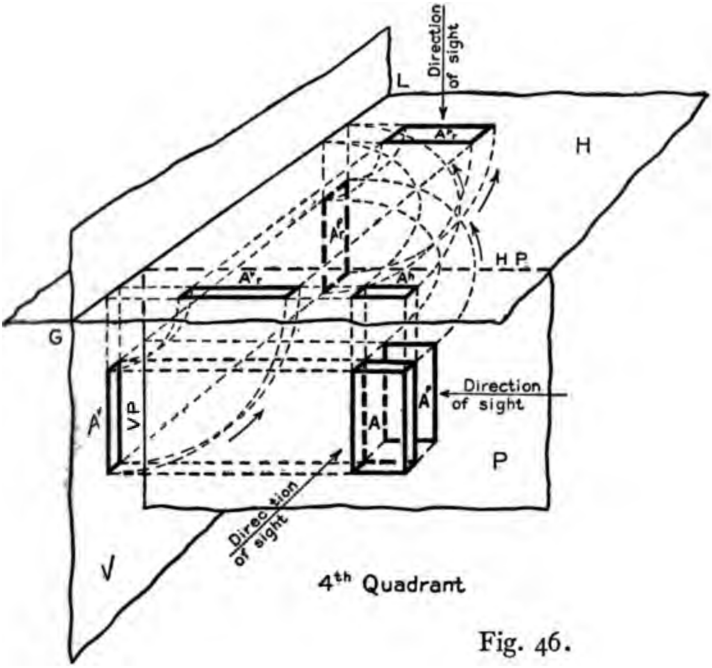
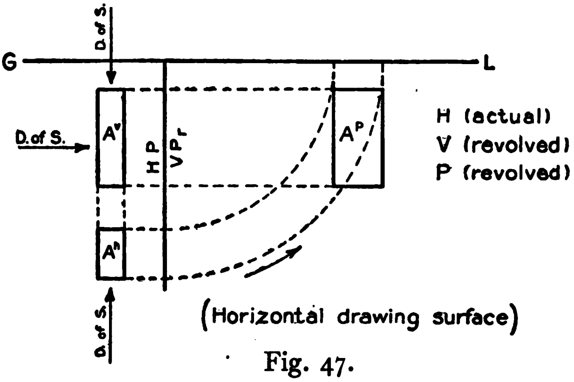


Fig. 46.



(Horizontal drawing surface)

Fig. 47.

H is between the eye and the object.
The object is between the eye and *V*.
The object is between the eye and *P*.
V is revolved into *H*.
P is revolved, first into *V*, and then into *H*.

CHAPTER III.

PROJECTIONS OF TYPICAL POINTS, LINES, PLANE FIGURES, AND SOLIDS; TRACES OF PLANES.

19. In visualizing a projection, it may be of assistance to see in imagination that either the thing projected covers one or more of its projections, or one or more of its projections covers the thing projected. It has been shown that these two conditions vary according to the quadrant used. Thus, for example, looking in the directions X, X', and X'', Fig. 48, the point in space, a, covers its projections a^h, a^v, and a^p. Likewise, looking in direction Y, Fig. 49, the point in space, b, covers its projection b^h; but, looking in directions Y' and Y'', the projections b^v and b^p cover the point in space b.

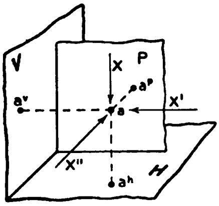


Fig. 48.

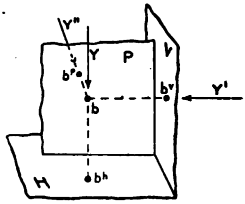


Fig. 49.

Visualize the following:—(*a*) The distance of a point from H must be measured in a perpendicular to H; such a distance can be shown only on a plane parallel to the direction of measurement; that is, on a plane, as V or P, perpendicular to H. The vertical projection of H is GL; distances from H, shown on V, must therefore be laid off from and perpendicular to GL.

(*b*) The distance of a point from V must be measured in a perpendicular to V; such a distance can be shown only on a plane parallel to the direction of measurement, that is on a plane, as H or P, perpendicular to V. The horizontal projection of *V* is *GL;* distances from *V* shown upon *H* must, therefore, be laid off from and perpendicular to *GL.*

(*c*) Distances from *H* shown on *P* revolved into *V* must be laid off from and perpendicular to *GL.* Distances from *H* shown on *P* revolved into *H* must be measured from and perpendicular to *HP.*

(*d*) Distances from *V* shown on *P* revolved into *V* must be laid off from and perpendicular to *VP.* If *P* is revolved into *H,* distances from *V* must be laid off from and perpendicular to *GL.*

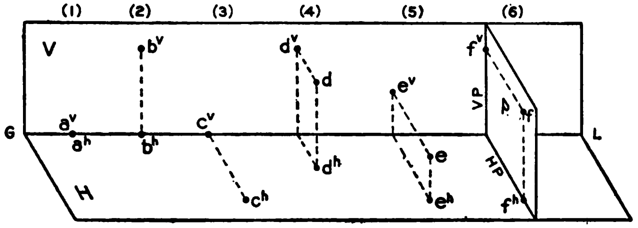


Fig. 50.

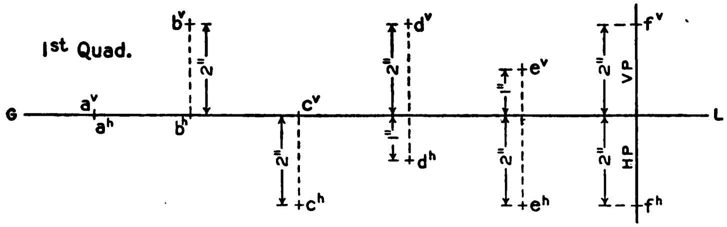


Fig. 51.

20. Alphabet of the Point.

(Figs. 50–54.)

Point *a,* common to *V* and *H*	(1)
" *b,* lying in *V,* 2" from *H*	(2)
" *c,* lying in *H,* 2" from *V*	(3)
" *d,* 2" from *H,* 1" from *V*	(4)
" *e,* 1" from *H,* 2" from *V*	(5)
" *f,* lying in *P,* 2" from *V,* and 2" from *H*	(6)

Exercise 6. Visualize the points *a-f*, Figs. 51–54, proceeding as follows:—

Model 1. Fig. 52 (5). Looking in a direction perpendicular to *V*, *GL* is seen as the vertical projection of *H* actual. The projection e^v covers point *e* in space; e^v is 1″ below *GL;* therefore point *e* will be seen in imagination 1″ below *H.*

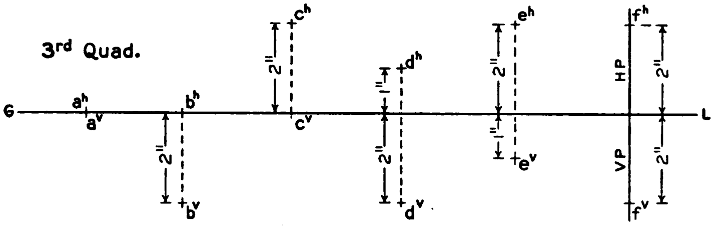


Fig. 52.

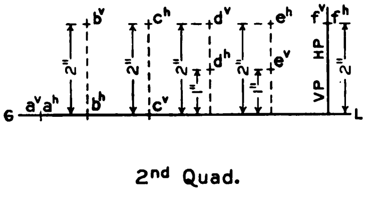


Fig. 53.

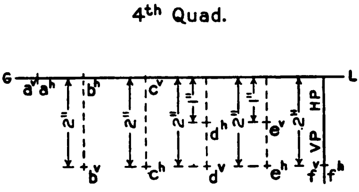


Fig. 54.

Looking in a direction perpendicular to *H*, *GL* is seen as the horizontal projection of *V* actual. The projection e^h covers the point *e* in space; e^h is 2″ behind *GL*, therefore the point *e* will be seen in imagination 2″ behind *V*. Combining the two distances, the point *e* is seen to lie in the third quadrant 1″ from *H* and 2″ from *V*. Or, looking in a direction perpendicular to *V*, if *H* is seen counter-revolved, the space projectors e^h and e^v may also be seen in imagination, and at their intersection the point *e* 1″ from *H* and 2″ from *V*.

Model 2. Fig. 54 (4). Looking in a direction perpendicular to *H*, *V* actual is seen edgewise and covering *GL*. The point *d* in space is covered by its projection d^h, and, as d^h is 1″ in front of *GL*, point *d* is seen 1″ in front of *V*. Looking in a direction perpendicular to *V*, *H* actual is seen edgewise and covering *GL*. The point *d* is seen covering d^v, and, as d^v is 2″ below *GL*, the point *d* is seen 2″ below *H*. Combining the two distances, it will be seen that the point *d* lies in the fourth quadrant, 2″ from *H* and 1″ from *V*.

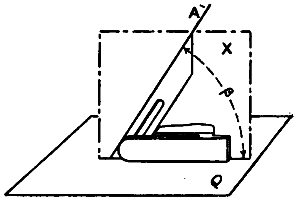


Fig. 55.

21. An angle, *β*, Fig. 55, between a line, *A*, and a plane, *Q*, must be measured in a plane, *X* (plane of angular measurement), containing *A* and perpendicular to *Q*.

22. Known and Unknown Quantities. When a projection shows an actual or *true* linear measurement, angle, area, or the shape of a plane figure, the projection in question may be called a *known quantity*. *Unknown quantities* are projections of *foreshortened* distances, angles, plane figures, and solids. The solution of problems includes the graphical finding of known quantities from unknown, and *vice versa*. In general, a *known quantity is a projection which is parallel to the line, angle, or surface projected.*

In Figs. 56–61 the known quantities are the projections showing the true lengths (TL), and true angles (TA)—with *H*, *V*, or *P*—of the lines represented. *The true length, and the true angle which a line makes with H, V, or P, is shown by the projection of the line on the plane or planes to which the line in space may be parallel.*

Visualize the following (no figure):—

(*a*) If, in measuring an angle between a line in space and *V*, the plane of measurement is parallel to *H*, then the angle between the horizontal projection of the line and *GL* is the same as the angle in the plane of measurement; the vertical projection of the plane of measurement is a line parallel to *GL*.

If, in measuring an angle between a line in space and *H*, the plane of measurement is parallel to *V*, then the angle between the vertical projection of the line and *GL* is the same as the angle in the plane of measurement; the horizontal projection of the plane of measurement is a line parallel to *GL*.

(*b*) The vertical projection of *H* actual is *GL*. If the vertical projection of a line in space is parallel to *GL*, the line must be parallel to *H;* the angle between the horizontal projection of the line and *GL* is the same as the angle between the line in space and *V*.

The horizontal projection of *V* actual is *GL;* if the horizontal projection of a line is parallel to *GL*, the line in space must be parallel to *V;* the angle between the vertical projection of the line and *GL* is the same as the angle between the line in space and *H*.

(*c*) When one projection of a line is a point, the line must be perpendicular to the corresponding co-ordinate plane; the line is, therefore, parallel to the other co-ordinate planes, and a projection of the line on either of these planes shows its true length.

(*d*) When a line inclined to *H* and *V* lies in a plane perpendicular to *H* and *V*, the horizontal and vertical projections of the line lie in the same ruled projector perpendicular to *GL;* hence these projections are indeterminate unless found by means of a profile projection (see Cases 13 and 14, Fig. 59).

(*e*) The vertical projection of *P* is *VP*, its horizontal projection is *HP*. If the horizontal projection of a line making an angle with *V* is parallel to *HP*, then the line in space is parallel to *P*, and the actual angle between the line and both *V* and *H* is shown on *P*. When the profile projection of a line is revolved into *V*, then the angle which it makes with *VP*, Fig. 59, is the same as the angle between the line in space and *V*. The angle between the *GL* and profile projection revolved into *V* is the same as the angle between the line in space and *H*. If, on the other hand, *P* be revolved into *H*, then the angle between *HP* and the profile projection of a line is the same as the angle between the line in space and *H*. The angle between the same projection and *GL* is the same as the angle between the line in space and *V*.

23. Alphabet of the Straight Line.

(FIGS. 56–61.)

Line A, lying in *V*, and perpendicular to *H* . . . . . (1)
" *B*, " " *H*, " " " *V* . . . . . (2)
" *C*, perpendicular to and ½″ from *H*, 2″ from *V* . (3)
" *D*, " " " " 1″ " *V*, 3″ " *H*. (4)

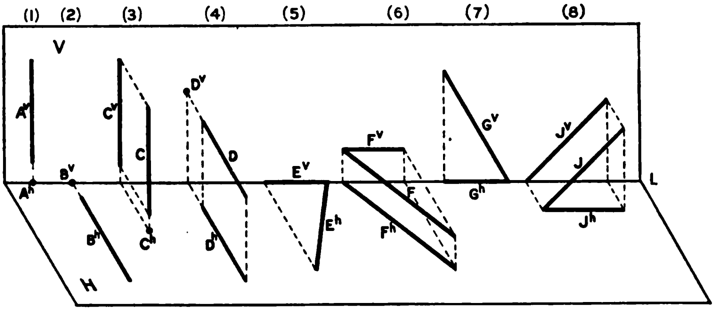


Fig. 56.

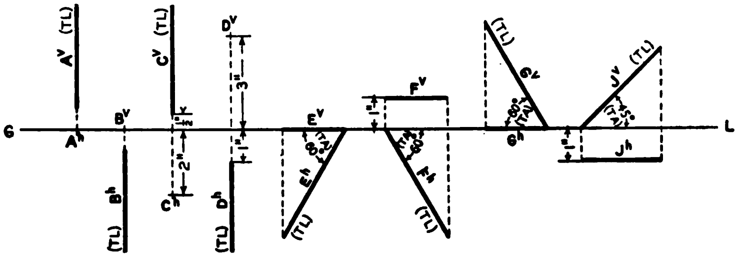


Fig. 57.

Line E, lying in *H*, touching *V*, and inclined backward to the right at 60° with *V* . . . . (5)

Line F, parallel to and 1″ from *H*, touching *V*, and inclined backward to the left at 60° with *V* . . . . (6)

Line G, lying in *V*, touching *H*, and inclined upward to the left, at 60° with *H* . . . . (7)

Line J, parallel to and 1″ from *V*, touching *H*, and inclined upward to the right at 45° with *H* . . . . (8)

Line *K*, lying in *V* and *H* . . . . (9)
" *L*, lying in *V*, parallel to and 3″ from *H* . . . . (10)
" *M*, " " *H*, " " " " " *V* . . . . (11)
" *N*, parallel to and 2″ from *H* and *V* . . . . (12)

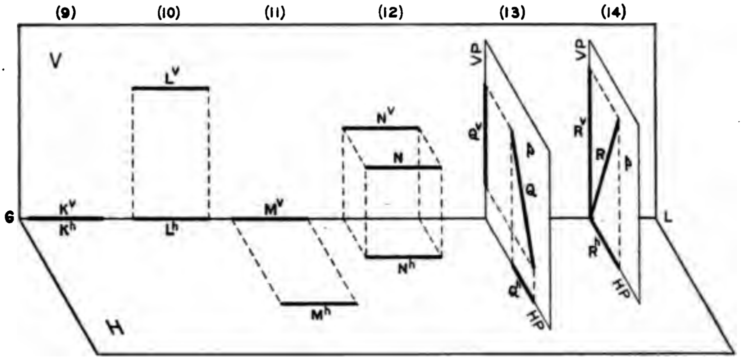


Fig. 58.

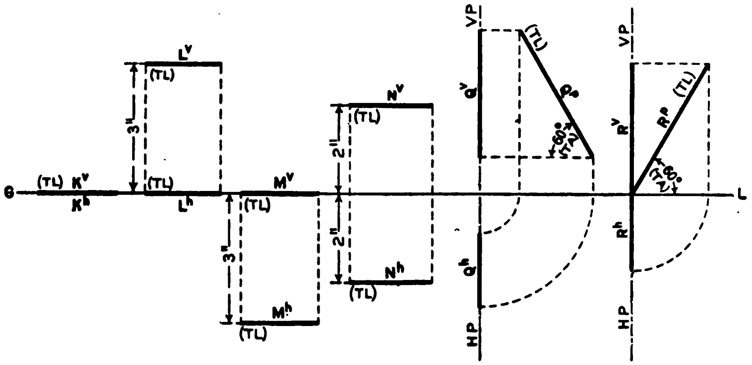


Fig. 59.

Line *Q*, inclined upward and backward at 60° with *H*, and lying in a plane perpendicular to *H* and *V* . . . . . (13)

Line *R*, inclined downward and backward at 60° with H, touching *GL*, and lying in a plane perpendicular to *H* and *V* . . . . (14)

Line S, touching *H* at a point 1″ from *V*, and inclined upward, forward, and to the right. (The horizontal projection is drawn at 45° with *GL*, the vertical projection at 60° with *GL*) . . . . (15)

Line T, inclined downward, backward, and to the right, touching *GL*. (Both projections are drawn at 45° with *GL*) . . . . (16)

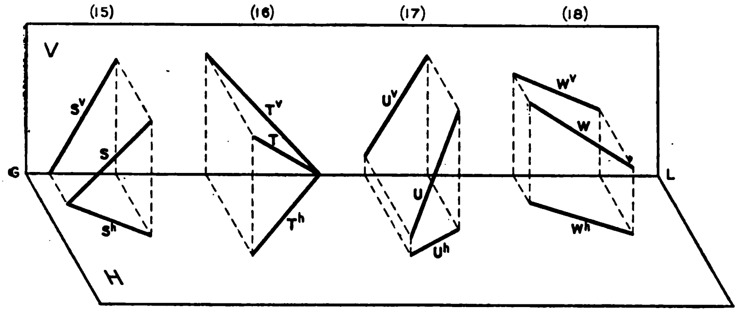


Fig. 60.

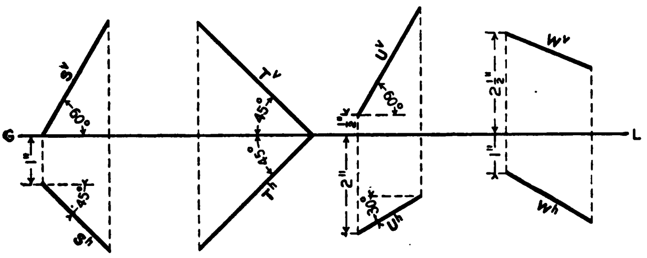


Fig. 61.

Line U, starting from a point ½″ from *H* and 2″ from *V*, inclined upward, backward, and to the right. (The horizontal projection is drawn at 30° with *GL*, the vertical projection at 60° with *GL*) . . . . (17)

Line W, starting from a point 2½″ from *H* and 1″ from *V*, inclined downward, forward, and to the right . . . . (18)

Exercise 7. Visualize, according to the following models, the preceding lines A to W (see also Figs. 56–61), *and note the known quantities (linear distances, and angles).*

Model 1. Fig. 57 (8). Looking in a direction perpendicular to *H*, *V* actual is seen edgewise and covering *GL*. The line *J* is seen in imagination covering J^h, and parallel to *V*. Looking in a direction perpendicular to *V*, *H* actual is seen edgewise and covering *GL*. The line *J* is seen covering J^v, and inclined upward to the right from *H*. Finally, the line *J* is seen in imagination covering at the same time both J^h and J^v and in its relation to both *H* and *V*.

Model 2. Fig. 59 (14). Looking in a direction perpendicular to *H*, the planes *V* and *P* actual are seen edgewise and covering *GL* and *HP* respectively. The line *R* is seen covering R^h and lying in *P*. Looking in a direction perpendicular to *V*, the planes *H* and *P* actual are seen edgewise and covering *GL* and *VP* respectively. The line *R* is seen covering R^v and lying in *P*. Looking in a direction perpendicular to *P* revolved, the planes *H* and *V* actual are seen edgewise and covering *GL* and *VP* respectively. The line *R* is seen coinciding with R^p and in its relation to *H* and *V*. Finally, seeing in imagination the three projections of *R* simultaneously, the line *R* is seen in its relation to the three co-ordinate planes.

Exercise 8. Make an instrumental drawing, using the given measurements, showing the projections of the lines *A* to *W* (Art. 23) in the *third quadrant.* Assume all measurements not given.

Line *G*, change "upward" to downward.
" *J*, " " " "
" *S*, " "upward, forward," to downward, backward.
" *U*, " "upward" to downward.
" *W*, " "downward, forward," to upward, backward.

Exercise 9. Draw the projections of the lines *A* to *W* lying in the *fourth quadrant.*

Line *G*, change "upward" to downward.
" *J*, " " " "

Line *Q*, change "backward" to forward.
" *R*, " "backward" to forward.
" *S*, " "upward" to downward.
" *T*, " "backward" to forward.
" *U*, " "upward" to downward.
" *W*, " "downward" to upward.

Exercise 10. Draw the projections of the lines *A* to *W* lying in the *second quadrant.*

Line *Q*, change "backward" to forward.
" *R*, " "downward" to upward.
" *S*, " "forward" to backward.
" *T*, " "downward" to upward.
" *U*, " "backward" to forward.
" *W*, " "forward" to backward.

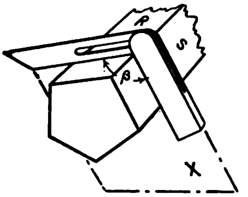


Fig. 62.

24. An angle, β, between two planes, *R* and *S*, Fig. 62, must be measured in a plane, *X*, (plane of angular measurement) perpendicular to each of the given planes. In order that a projection shall show a true angle between planes, the plane of projection must be parallel to the plane of angular measurement.

25. Alphabet of Plane Figures and Planes.

(Figs. 63–68.)

Plane figure Q, perpendicular to *H* and *V*. The H- and V-projections are straight lines perpendicular to *GL*. Each projection shows one true dimension of the figure, but to show its true shape and area requires a P-projection (Fig. 64). The traces of the indefinite plane containing the figure contain the H- and V-projections respectively . . . . . . (1)

Plane figure R, parallel to *H*. The H-projection shows the true shape and area of the figure. The V-projection is a straight line parallel to *GL*. The V-trace of the indefinite plane containing the figure contains the V-projection. The plane has no H-trace . . . . (2)

Plane figure S, parallel to *V*. The V-projection shows the true shape and area of the figure. The H-projection is

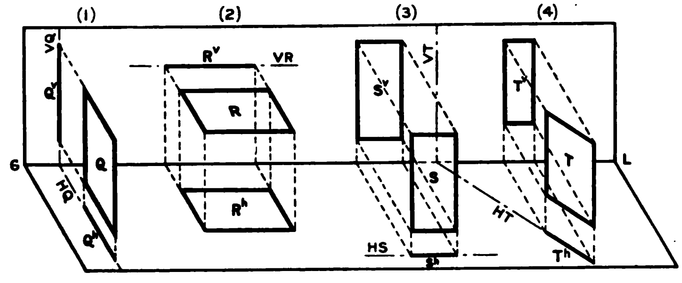


Fig. 63.

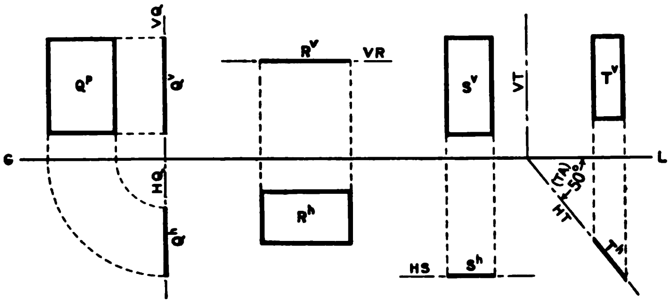


Fig. 64.

a straight line parallel to *GL*. The H-trace of the indefinite plane of the figure contains the H-projection. The plane has no V-trace . . . . (3)

Plane figure T, perpendicular to *H*, and making an angle, backward to the left, of 50° with *V*. The V-projection shows the figure foreshortened. The plane of angular measurement (not shown in Fig. 63) is perpendicular to the figure and to *V*, hence the true angle which the figure makes with *V* is shown by the H-projection. The H-trace of the indefinite plane of the figure contains the H-projection. The V-trace is perpendicular to *GL*. The traces must intersect in *GL* . . (4)

Plane figure U, perpendicular to *V*, and making an angle, downward to the left, of 45° with *H*. The plane of angular measurement (not shown) is perpendicular to the figure and

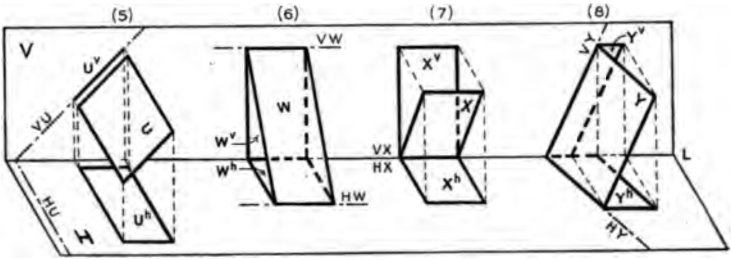


Fig. 65.

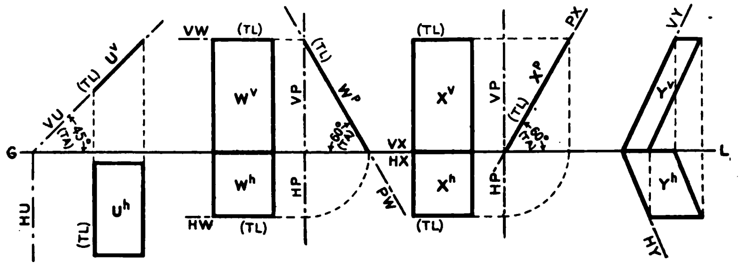


Fig. 66.

to *H*, hence the true angle which the figure makes with *H* is shown by the V-projection. The V-trace of the indefinite plane of the figure contains the V-projection; the H-trace is perpendicular to *GL* . . . . (5)

Plane figure W, parallel to *GL*, touching *H* and *V*, and inclined at 60° to *H*. The H- and V-projections show the figure foreshortened. Since the figure is parallel to *GL*, it is perpendicular to *P*, hence its P-projection shows the true angle between the figure and *H*. The H- and V-traces of the indefinite plane of the figure are parallel to *GL.* The P-trace contains the P-projection . . . . (6)

Plane figure X, containing a portion of *GL,* and making an angle of 60° with *H.* The H- and V-projections show the figure foreshortened. Since one edge of the figure lies in *GL,* the figure is perpendicular to *P,* hence the P-projection shows the true angle between the figure and *H.* The H- and V-traces of the indefinite plane of the figure lie in *GL.* The P-trace contains the P-projection . . . . (7)

Plane figure Y, inclined to *H, V,* and *P,* and sloping upward, backward, and to the right . . . . (8)

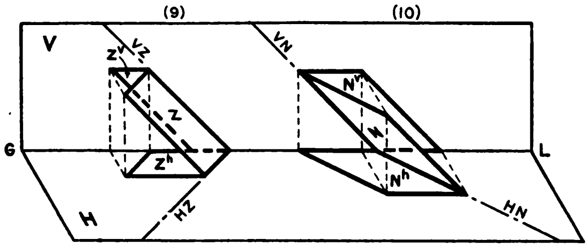


Fig. 67.

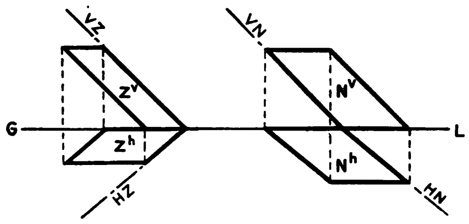


Fig. 68.

Plane figure Z, inclined to *H, V,* and *P,* and sloping upward, backward, and to the left . . . . (9)

Plane figure N, inclined to *H, V,* and *P,* and sloping upward, forward, and to the left . . . . (10)

Exercise 11. Make a pencil drawing showing in the *third quadrant:* (*a*) The projections of the plane figures 1–7. (*b*) The traces (for all quadrants crossed by each plane) of the indefinite planes of the figures 1–10. Assume all measurements except the stated angles. Letter, according to the notation (Art. 17), each projection and trace. Indicate each known quantity by the appropriate use of TL or TA.

Exercise 12. Make a drawing, according to the instructions in Exercise 11, showing the plane figures and planes in the *second quadrant.*

26. Some Typical Solids.

FIG. 69.

A square right prism (2). Placed in the third quadrant; the nearest long edge is perpendicular to *H*, and $\frac{3}{8}''$ from *V;* the faces are inclined at 45° to *V*. The prism is cut by a plane, *X*, perpendicular to *V*, inclined at 45° to *H*, and intersecting the lower base $\frac{5}{16}''$ from the right-hand corner. The H-projection of the section is invisible.

A square right prism (3). The preceding placed in the second quadrant; the nearer base lies in *V*, and the lowest long edge in *H;* the left-hand lower face is inclined at 30° to *H*. The prism is cut by a plane, *X*, parallel ‸to H and $\frac{13}{16}''$ from *V*. The H-projection of the section is visible.

A triangular right prism (4). Placed in the fourth quadrant; the bases are parallel to *H*, the upper one $1\frac{7}{16}''$ from *H*. The left-hand corner of the upper base is $\frac{5}{8}''$ from *V;* the nearest edge of the same base is $1\frac{3}{16}''$ long, and inclined at 30° to *V;* the length of the right-hand edge of the base is $1\frac{1}{8}''$, and that of the farthest edge $\frac{7}{8}''$. The prism is intersected by a plane, *X*, parallel to *V*, and $\frac{7}{16}''$ from the vertical front edge of the solid.

A hexagonal right pyramid (9). Placed in the first quadrant; the base lies in *H*, its centre $\frac{7}{8}''$ from *V;* two basal edges are parallel to *V*. The pyramid is cut by a plane, *X*,

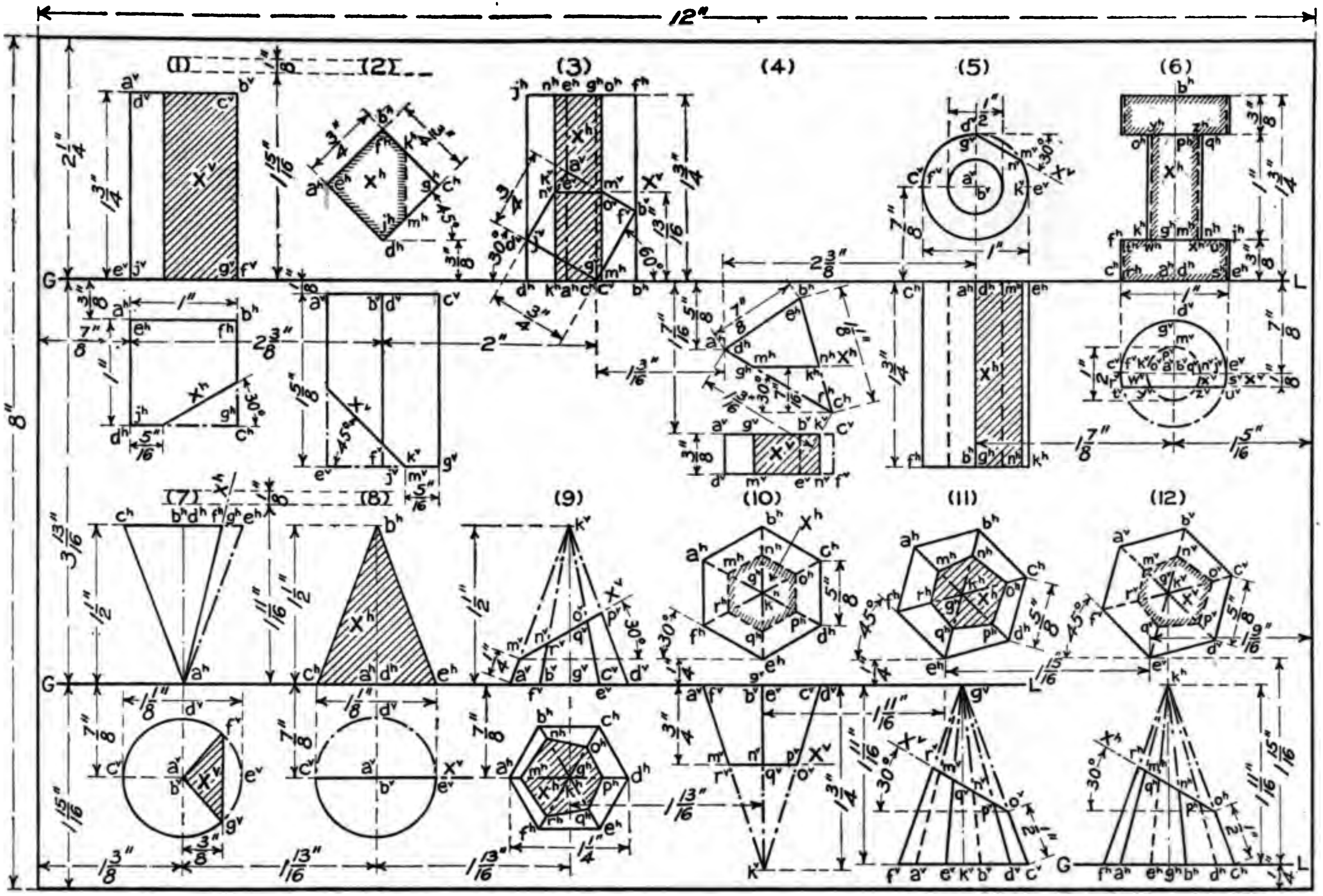

12"
8"
(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)
(9)
(10)
(11)
(12)
G
L
X^v
X^h
$2\frac{1}{4}$"
$1\frac{3}{4}$"
$1\frac{5}{16}$"
$2\frac{3}{8}$"
2"
$3\frac{13}{16}$"
$1\frac{3}{8}$"
$1\frac{13}{16}$"
$1\frac{7}{8}$"
30°
45°
60°

Fig. 69.

perpendicular to *V*, inclined at 30° to *H*, and intersecting the extreme left-hand inclined edge of the solid ¼″ from *H*. The H-projection of the section is visible.

Exercise 13. Make an instrumental pencil drawing of the plate, Fig. 69, according to the given measurements. Visualize all *lines*, *planes*, and *solids* as the drawing proceeds. Note which edges of the solids and of the sections are ***invisible.*** Letter the projections, but omit all measurements.

Exercise 14. Describe completely, in writing, the solids and their positions, etc., except those described above.

27. Solids Projected on H, V, and P.

(FIG. 70.)

Exercise 15. Make a pencil drawing in connection with the following text and according to the measurements in Fig. 70. Visualize each step as the drawing proceeds, giving particular attention to the visibility of all edges of the solids. Letter the projections,—not merely copying the letters from the plate, but as a means of accounting for all points, lines, and surfaces of the solids in space. Do not give the dimensions.

A hexagonal right prism (13). Placed in the third quadrant, the right-hand base lying in *P*. Two lateral faces are parallel to *H*, and one lateral edge is ⅞″ from *H*, ⅜″ from *V*. ***Construction.*** Since the bases are perpendicular to *H* and *V*, the H- and V-projections of the prism must be found from its P-projection. Consider *P* as revolved into *V;* draw the P-projection, and find the V-projection. Counter-revolve the P-projection, and find the H-projection.

A square right pyramid (14). Placed in the first quadrant, to the left of *P*. The base is parallel to *P;* the apex lies in *P*. The lowest corner of the base is ¼″ from *H*, 1⅛″ from *V;* from this point one edge of the base inclines upward and forward at 60° to *H*. The pyramid is cut by a plane *X*, perpendicular to *H*, making 60° with *V*, and intersecting the nearest lateral edge of the pyramid ⅝″ from the base. *Construction.* Since

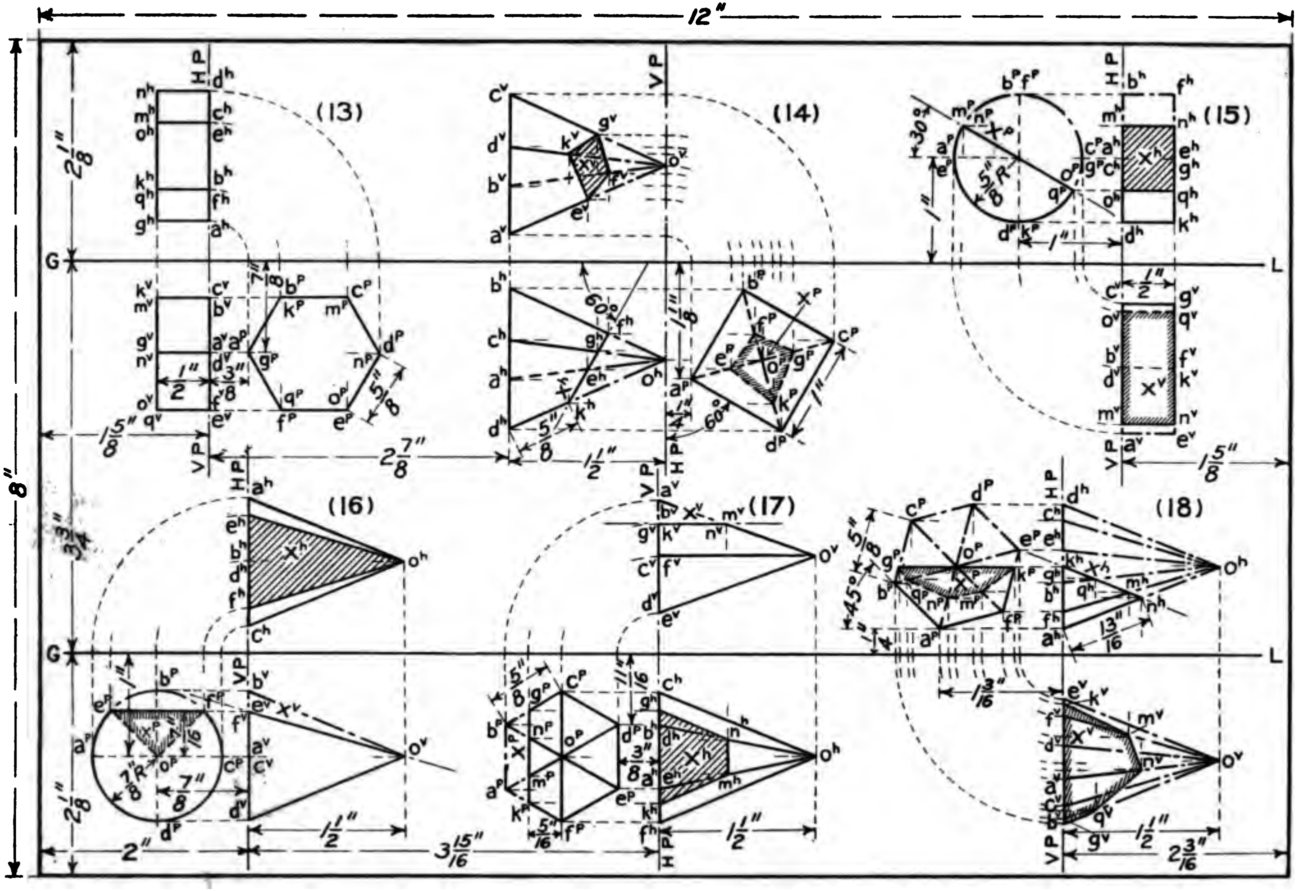

12"
8"
(13)
(14)
(15)
(16)
(17)
(18)
G
L
H P
V P
$2\frac{1}{8}$"
$3\frac{3}{4}$"
$2\frac{7}{8}$"
$1\frac{5}{8}$"
$1\frac{1}{2}$"
$3\frac{15}{16}$"
$2\frac{3}{16}$"
2"
60°
45°
30°

Fig. 70.

the base is perpendicular to *H* and *V*, the H- and V-projections of the pyramid must be found from its P-projection. Consider *P* as revolved into *H;* draw the P-projection, and find the H-projection. Counter-revolve *P*, and find the V-projection. Find the projections of the section in *X;* check the accuracy of the intersections by counter-revolving the P-projection of the section, as shown.

A right cylinder (15). Placed in the third quadrant, the left-hand base lying in P, its centre 1″ from *H* and 1″ from *V*. The cylinder is cut by a plane *X*, perpendicular to *P*, making an angle of 30° with *V*, and passing through the centres of the bases. Find the H-, V-, and P-projections of the cylinder and of the section in *X*. Revolve *P* into *H*.

A right cone (16). Placed in the third quadrant to the right of *P*, the base lying in *P*, its centre 1″ from *H* and $\frac{7}{8}$″ from *V*. The cone is cut by a plane, *X*, perpendicular to *V*, passing through the apex, and cutting the base in a line parallel to *H*, $\frac{7}{16}$″ from the centre of the base. Find the H-, V-, and P-projections of the cone and of the section in *X*.. Revolve *P* into *V*.

A hexagonal right pyramid (17). Placed in the first quadrant, to the right of *P*, the base lying in *P*. Two basal edges are parallel to *H;* the lower of these is $\frac{3}{8}$″ from *H*, and its further end is $\frac{11}{16}$″ from *V*. The pyramid is cut by a plane, *X*, parallel to *H*, and intersecting the base $\frac{5}{16}$″ above its centre. Find the H-, V-, and P-projections of the pyramid and of the section in *X*. Revolve *P* into *H*.

A hexagonal right pyramid (18). Placed in the third quadrant, to the right of *P*, the base lying in *P*. From a point $1\frac{3}{16}$″ from *H*, $\frac{1}{4}$″ from *V*, one edge of the base is inclined downward and backward at 45° to *V*. The pyramid is cut by a plane, *X*, perpendicular to *H*, passing through the centre of the base, and intersecting the nearest lateral edge of the pyramid at a point $\frac{13}{16}$″ from the base. Find the H-, V-, and P-projections of the pyramid and of the section in *X*. Revolve *P* into *H*.

CHAPTER IV.

THEOREMS; PROBLEMS ON THE RELATIONS BETWEEN RIGHT LINES AND THE CO-ORDINATE PLANES.

28. *If two lines in space are parallel, their projections on any plane are parallel.*

Let A and B, Fig. 71, be parallel. The plane projectors AA^v and BB^v are parallel, since each, by definition, is perpendicular to a plane of projection, V, and contains one of two parallel lines. The intersections, A^v and B^v, of the plane projectors with V are therefore parallel. (If two parallel planes are cut by a third plane, the lines of intersection are parallel.) Similarly, the plane projectors AA^h and BB^h are parallel. Hence A^h_r and B^h_r are parallel. (See also Fig. 72.)

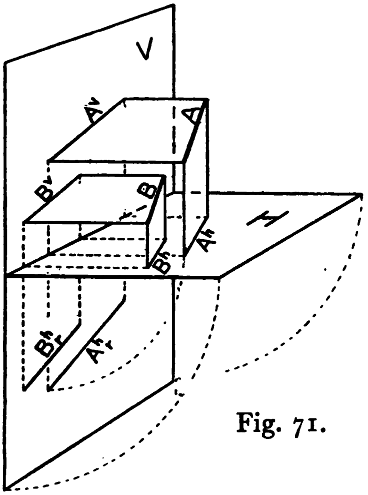


Fig. 71.

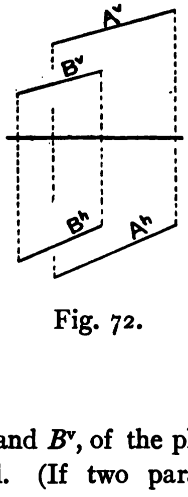


Fig. 72.

Conversely, if *both* the vertical and horizontal projections of two lines not parallel to P are parallel, the lines in space are parallel.

29. *If two lines in space intersect, their projections must intersect in a point lying in the same ruled projector perpendicular to GL.*

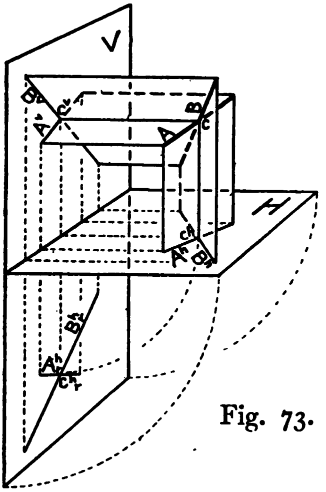


Fig. 73.

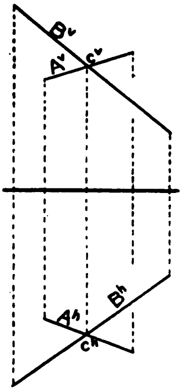


Fig. 74.

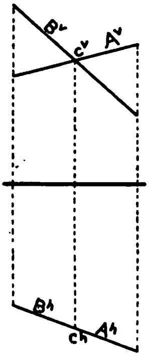


Fig. 75.

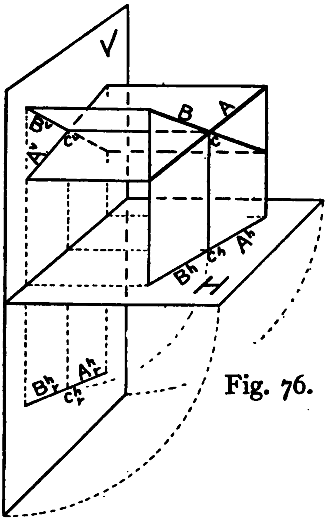


Fig. 76.

(*a*) Let *A* and *B*, Fig. 73, intersect in the point *c*. The plane projectors AA^v and BB^v, perpendicular to *V*, intersect in the line cc^v, perpendicular to *V*. The plane projectors

AA^h and BB^h, perpendicular to H, intersect in the line cc^h, perpendicular to H. But cc^v and cc^h are the line projectors of the point c, which must fall in the ruled projector, $c^vc^h{}_r$, perpendicular to GL. (See also Fig. 74.)

(*b*) In Fig. 75 the intersecting lines A and B lie in the same plane projector perpendicular to H, hence the horizontal projections of the lines coincide. (See also Fig. 76.)

30. *If one of two lines forming a right angle is parallel to a plane of projection, the right angle must project as such (unless the second line projects as a point).*

Let A and B, Fig. 77, forming a right angle, be parallel to the plane of projection H. The projections A^h and B^h therefore form a right angle. Revolve the line B about line A; then the plane Z, containing B, B_1, B_2, etc., successive positions of B, will be perpendicular to A (all the perpendiculars drawn to a straight line at a given point lie in a plane perpendicular to the line). The plane Z, perpendicular to line A, is perpendicular to A^h, parallel to A, hence perpendicular to H. But the horizontal projection of the plane Z is a line, HZ, containing B^h; therefore any position of the line B in plane Z must project on H in B^h, drawn at right angles to A^h. (See also the projections, Fig. 78, in which the plane Z is taken at 45° with V.)

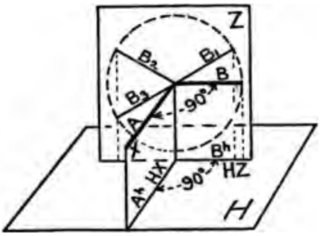


Fig. 77.

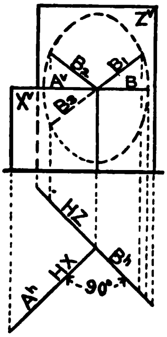


Fig. 78.

Problem 1. Given the projections of a line inclined to H and V, to find the true length of the line, and its true angles with H and V.

(*a*) *First Method.* Let a^vb^v and a^hb^h, Fig. 79, be the given projections. It is required to find the true length of *ab*, and the true angle which *ab* makes with *H*. About a vertical axis taken through the point *a*, revolve *ab* parallel to *V*. Then its vertical projection shows the true length of *ab* and the true angle with *H*.

Construction. With a^h as centre and a^hb^h as radius, describe the arc $b^hb_1^h$, which is the H-projection of the path of point *b* in space (Art. 16). Draw $a^hb_1^h$ parallel to *GL*. The vertical projection of the path of *b* is the line $b^vb_1^v$ parallel to *GL*. Project from b_1^h to $b^vb_1^v$, giving b_1^v, and draw $a^vb_1^v$, the required projection.

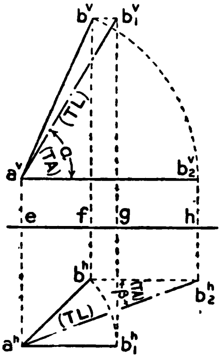


Fig. 79.

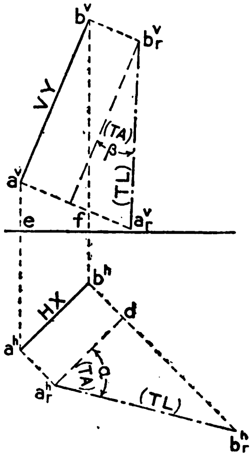


Fig. 80.

To find the true angle which the line *ab* makes with *V*, revolve *ab* about an axis taken through point *a* and perpendicular to *V*. The resulting H-projection, $a^hb_2^h$, shows the true length of the line and its true angle, β, with *V*.

(*b*) *Second Method.* By definition, the plane projector of a line is perpendicular to a co-ordinate plane. If the plane projector be revolved into the co-ordinate plane, the true length of the line projected by the plane will be shown, and also the true angle which the projected line makes with the co-ordinate plane into which it is revolved.

Construction. Let ab, Fig. 80, be the given line, X its plane projector to H, and Y its plane projector to V.

Let plane X be revolved into H. Draw perpendiculars at the extremities of a^hb^h. Make $a^ha_r^h$ equal to ea^v, and $b^hb_r^h$ equal to fb^v. Connect a_r^h and b_r^h, which gives the true length of ab. The true angle which ab makes with H is α.

Let the plane projector Y be revolved into V. Draw perpendiculars at the extremities of a^vb^v. Make $a^va_r^v$ equal to ea^h, and $b^vb_r^v$ equal to fb^h. Connect a_r^v and b_r^v which gives the true length of ab. The true angle which ab makes with V is β.

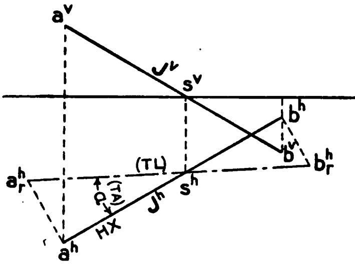


Fig. 81.

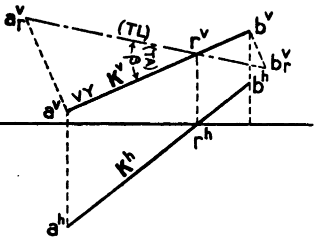


Fig. 82.

In Fig. 81 the line, J, passes from the first into the fourth quadrant. The plane projector of the portion as of J revolves about the trace HX downward into H, while the portion sb revolves upward into H; hence the revolved positions of a and b fall on *opposite* sides of the trace HX.

Exercise 16. In Fig. 82 the line K is revolved into V. Compare with Fig. 81, and visualize the construction.

Problem 2. Given the true angle which a line makes with one co-ordinate plane, and the angle which the projection on that plane makes with GL, to draw the projections of the line.

(*a*) Let it be required to draw the projections of a line making an angle of 60° with *H*, and having its horizontal projection placed at 45° with *GL*.

If a line in space *ab*, Fig. 83, is perpendicular to *H*, its horizontal projection is a point, $a^h b^h$, and the line in question is parallel to *V*. Let the line be inclined to *H* at the given angle, 60°. In order that the angle which a line in space makes with *H* shall be shown on *V*, it is necessary that the line be parallel to *V;* that is, that its horizontal projection, $a_1^h b_1^h$, be parallel to *V*. Placing $a^v b^v$ at 60° with *H*, as shown at $a_1^v b_1^v$, and drawing $a_1^h b_1^h$ parallel to *V*, the length of $a_1^h b_1^h$ may be found by drawing vertical projectors from a_1^v and b_1^v.

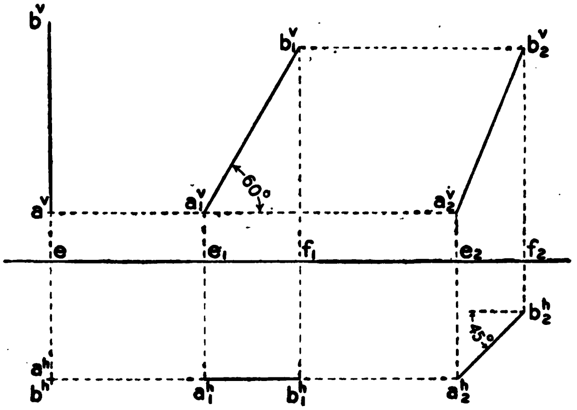


Fig. 83.

It will be seen that the length of $a_1^h b_1^h$ is a fixed quantity for a line of given length and making a given angle with *H*. If the horizontal projection be placed in any new position, making an angle with *GL*, it will follow that the line in space is no longer parallel to *V*. The angle which the line in space makes with *H*, however, remains the same (60°), since the length of its *horizontal* projection remains the same. Furthermore, the distances $a_1^ve_1$ and $b_1^vf_1$ will not change.

Place $a_1^hb_1^h$ at 45° with *GL*, as shown by $a_2^hb_2^h$. To find the vertical projection of *ab* for this new position, project horizontally from $a_1^vb_1^v$ and vertically from $a_2^hb_2^h$, giving $a_2^vb_2^v$, the required projection.

Observe that both $a_2^hb_2^h$ and $a_2^vb_2^v$ are unknown quantities as regards linear measurement; also that the line in space now makes *some* angle with *V*, but this angle is *not* 45°.

(*b*) The construction, Fig. 83, is given in different form in Fig. 84. Draw the projectors of *ab* perpendicular to *H*. Revolve *ab* about an axis through point *a*, and perpendicular to *V*, until inclined at 60° with *H*. In this revolution the line *ab* moves in a plane perpendicular to the axis of revolution; that is, the point *b* moves in a circular arc parallel to *V;* the horizontal projection of this arc is $a^hb_1^h$ parallel to *V*. Now swing *ab* about an axis through point *a* and perpendicular to *H*, until the horizontal projection of the line in space makes an angle of 45° with *V*. In this revolution, point *b* moves in a circular arc parallel to *H;* the vertical projection of this arc is $b_1^vb_2^v$ parallel to *H*. Draw $a^vb_2^v$, the resultant projection of *ab*.

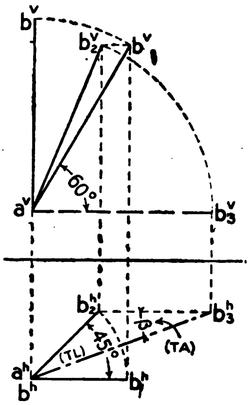


Fig. 84.

(*c*) To find the angle which *ab* in its last position makes with *V*, swing $a^vb_2^v$ about an axis through *a* and perpendicular to *V*, until parallel to *H*. In this revolution b_2^v moves in a circular arc parallel to *V;* the horizontal projection of this arc is $b_2^hb_3^h$ parallel to *V*. Connect $a^hb_3^h$, giving the horizontal projection of *ab* parallel to *H*, and showing its true angle with *V*.

Exercise 17. Draw Figs. 85 and 86. State each problem, and then describe in each: (*a*) the steps by which the projections

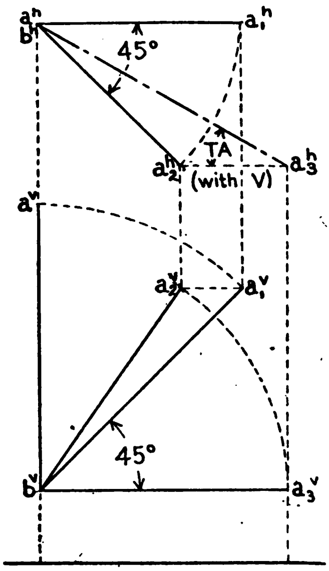


Fig. 85.

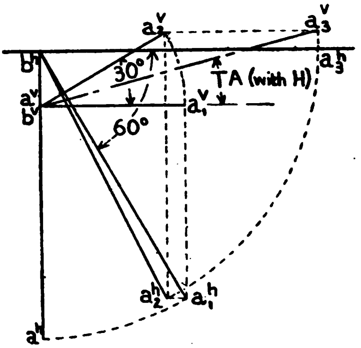


Fig. 86.

of *ab* are found; (*b*) the steps by which the true angle with *H* or *V* is obtained.

Problem 3. To draw the projections of a line making given angles with V and H.

Let *ab*, Figs. 87 and 88, be the given line, the given angles 30° with *V* and 45° with *H*. Take *ab* of some definite length, and with either end at any assumed point, as *a*. Place *ab* (ab_1, Fig. 88) parallel to *H*, making the given angle of 30° with *V*, as shown by its projections $a^h b_1^h$ and $a^v b_1^v$, Fig. 87. Taking *ab* (ab_2, Fig. 88) the *same* length, place it parallel to *V*, making the given angle of 45° with *H*, as shown by its projections $a^v b_2^v$ and $a^h b_2^h$, Fig. 87. Revolve *ab* about an axis, through point *a*, perpendicular to *V*. In this revolution the point *b* moves in the circular arc X (X^v, X^h). Next, revolve *ab* about an axis, through point *a*, perpendicular to H. The path of point *b* is the circular arc Y (Y^v, Y^h). Since every point in arcs X and Y is equally distant from point *a*, they must intersect in some point, as *b* (b^v, b^h). Then *ab* (a^vb^v, a^hb^h) is the required line, because, if a straight line be drawn to a^v from any point in the arc X^v, since it is equal to $a^vb_1{}^v$, it must represent *ab* in space making 30° with V. Also, if a straight line be drawn to a^h from any point in the arc Y^h, since it is equal to $a^hb_2{}^h$, it must represent *ab* making 45° with H. But a^hb^h is the projection not only of *ab* inclined at 45° to H, but also of the same line at 30° with V, since the point b^h is common to the arc Y^h and the straight line X^h, the horizontal projection of the path X. A similar proof applies to the projection a^vb^v.

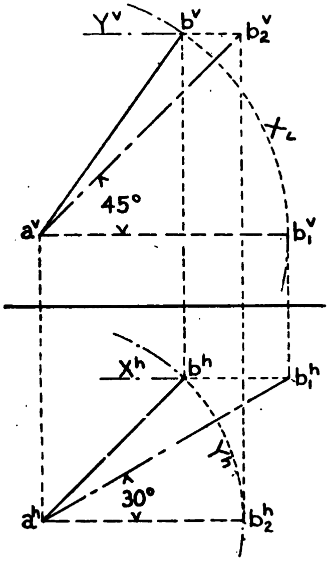


Fig. 87.

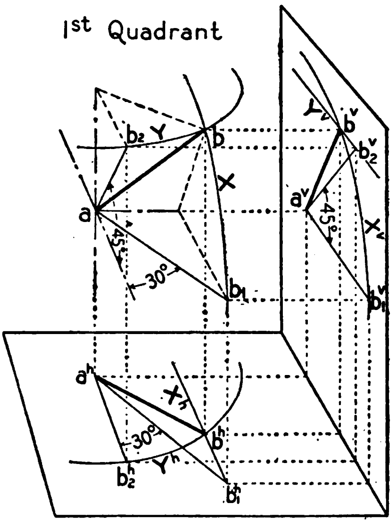


Fig. 88.

Exercise 18. Draw the projections of a line 2″ long, lying in the second quadrant, and inclined upward, backward and to the right. The line makes 45° with H, 15° with V. Place the front end of the line 1⅛″ from H, ⅜″ from V.

Exercise 19. (*a*) Draw the projections of a line 2″ long, placed in the third quadrant, inclined upward, backward, and to the left. The line makes 30° with *H* and 45° with *V*. Its lower end is 2¼″ from *H* and ½″ from *V*.

(*b*) Draw the projections of the same line placed in the second quadrant.

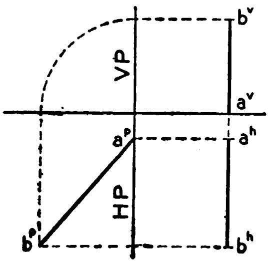


Fig. 89.

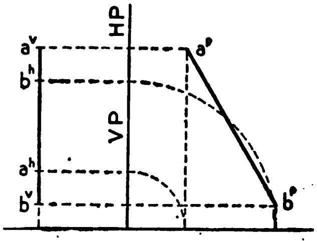


Fig. 90.

Special Case. When the sum of the given angles with *H* and *V* is 90°, the required line is parallel to *P*. Construct

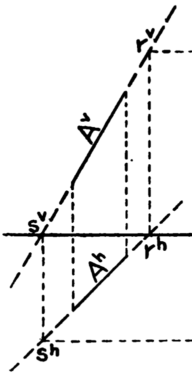


Fig. 91.

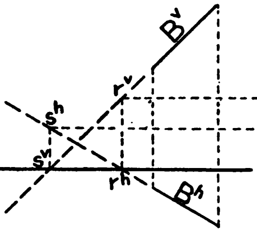


Fig. 92.

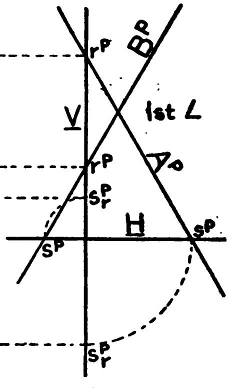


Fig. 93.

first the profile projection (Figs. 89 and 90), showing the line in its true relations to *H* and *V*, then find the horizontal and vertical projections by counter-revolving *P*.

Problem 4. Given the projections, to find the traces of a straight line.

Let A^v, A^h, Fig. 91, be the given projections. Vizualizing A, it is seen to lie in the first quadrant, and to incline upward, backward, and to the right. If A be produced upward, it must meet V; this will be shown by its horizontal projection intersecting V (seen edgewise) at r^h. The actual point r^v in which A pierces V must lie in A^v, and this point is obtained by projecting from r^h. If A be produced downward, it must meet H. This occurs when A^v meets H, as at s^v. The point s^h, in which A pierces H, lies in A^h, and is projected from s^v. Visualizing A, it is seen to pass from the first quadrant into the second and fourth quadrants.

The line B, Fig. 92, is in the first quadrant. Visualizing the line, it is seen to incline downward, backward, and to the left. Producing B^h and B^v, it will be observed that B^h meets V before B^v meets H; that is, B passes through V into the second quadrant. The fact that B^v meets H shows that B pierces H, and this trace is behind V. Visualizing B, the line is seen to pass from the first quadrant, through the second, into the third quadrant.

In Fig. 93 the preceding lines A and B are projected on P, V and H being revolved about a vertical axis until seen edgewise. To draw the profile projection of A, set off s^p from V equal to $s^v s^h$, Fig. 91, or project and revolve as shown. Set off r^p equal to $r^v r^h$, or project as shown. A^p, connecting s^p and r^p, shows clearly the line A passing through the fourth, first, and second quadrants. Draw B^p by the same method.

In Fig. 94 are shown the traces of line C. It passes through the fourth, third, and second quadrants (see also Fig. 96). The line D, Fig. 95, parallel to H and inclined to V, can have but one trace, r^v (see also Fig. 96).

Special Case. (*a*) Given the projections, $a^h b^h$ and $a^v b^v$, Fig. 97, of a line lying in the third quadrant and in a plane perpendicular to H and V. In order to find the traces of ab, it is necessary to use the profile plane. Having drawn $a^p b^p$, here revolved into *V*, produce this projection to intersect *H* in s^p and *V* in r^p. Counter-revolve s^p and r^p, giving the required traces s^h and r^v.

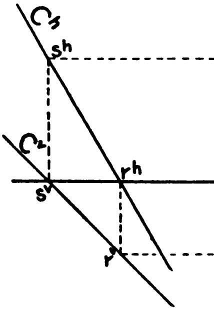


Fig. 94.

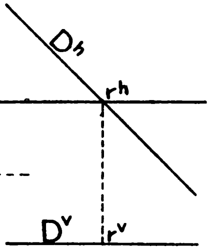


Fig. 95.

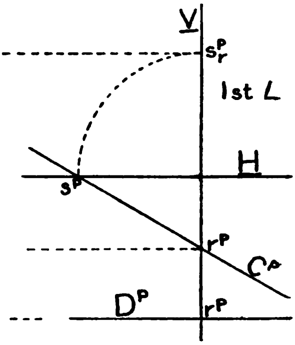


Fig. 96.

(*b*) Given the projections of a line *de*, Fig. 98, situated in the third quadrant. *P* is here passed through the given pro-

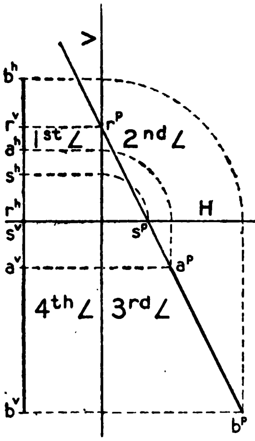


Fig. 97.

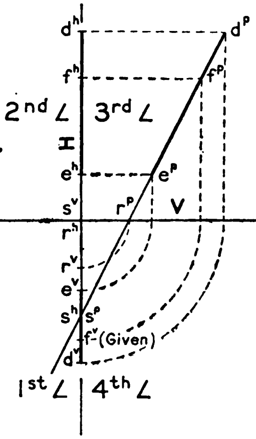


Fig. 98.

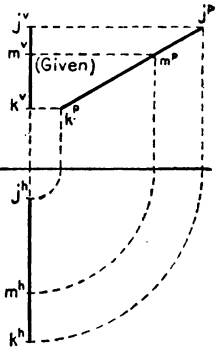


Fig. 99.

jections and revolved into *H*. Revolving *P* about its horizontal trace, *H* and *V* also revolve about the same line, with the result

Errata

that, when *P* is seen flatwise, *H* and *V* are seen edgewise, as shown by the lettering in Fig. 98. Produce d^pe^p to intersect *V* in r^p and *H* in s^p. Counter-revolve r^p and s^p, giving r^v and s^h, the required traces.

(*c*) Besides the traces, it is required to find the horizontal projection of the point *f*, Fig. 98, lying in *de*, having given its vertical projection f^v. Revolve f^v into d^pe^p, giving f^p, and counter-revolve f^p into d^he^h, giving f^h, its required projection.

(*d*) The profile line, *jk*, Fig. 99, lying in the first quadrant, is revolved into *V*. The vertical projection of point *m*, lying in *jk*, being given, it is required to find the horizontal projection of the point. Visualize all conditions.

Problem 5. Given the traces, to find the projections of a line.

Let r^v and s^h, Fig. 100, be the given traces. The horizontal projection of vertical trace r^v is r^h. Connect r^h and s^h, giving r^hs^h the required horizontal projection. The vertical projection of horizontal trace s^h is s^v. Connect r^v and s^v, giving r^vs^v the required vertical projection. Visualizing *rs*, it is seen to pass through the first quadrant into the second and fourth quadrants (see also the profile projection r^ps^p, Fig 102).

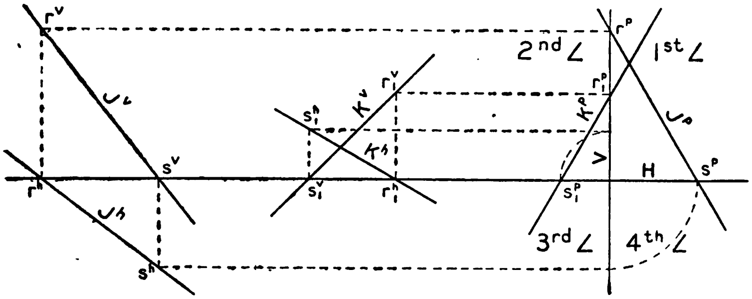


Fig. 100. Fig. 101. Fig. 102.

In the second example, Fig. 101, the line r_1s_1, piercing *V* and *H* in r_1^v and s_1^h respectively, passes through the second quadrant into the first and [illegible] quadrants.

Given the traces of a profile line, Fig. 103, lying in the third quadrant, also the horizontal projection of a point, *c*, lying in the line. It is required to find the vertical projection of *c*. The profile plane is passed through the given traces and revolved into *H*. Visualize all conditions, and decide into what quadrants *rs* will pass if produced.

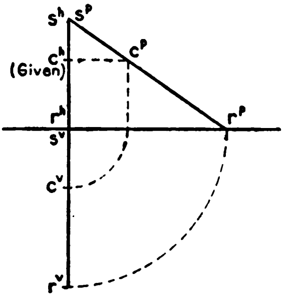


Fig. 103.

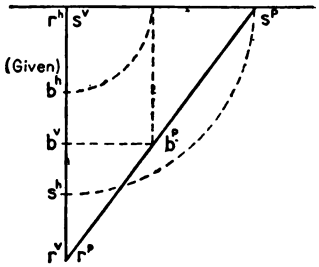


Fig. 104.

Given the traces of a profile line, Fig. 104, lying in the fourth angle, also the horizontal projection of a point, *b*, lying in the line. It is required to find the vertical projection of *b*. The profile plane is revolved into *V*. Visualize all conditions, and decide into what angles *rs* will pass if produced.

CHAPTER V.

REVOLUTION OF PLANES AND PLANE FIGURES; PROBLEMS ON THE LINE AND PLANE.

31. **Parallels of a Plane.** These are lines parallel to V or H, lying in a plane.

Let the plane X, Fig. 105, be intersected by the plane H_1, parallel to H; then the line of intersection M (an H-*parallel*) is parallel to the horizontal trace of X. (If two parallel planes are cut by a third plane, the lines of intersection are parallel.) It is evident that, as M is parallel to HX, the horizontal projection of M (M^h, Fig. 106) must be parallel to

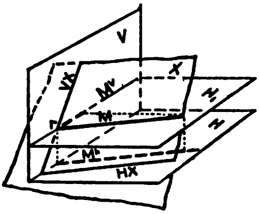


Fig. 105.

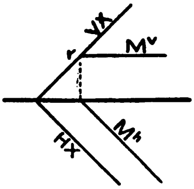


Fig. 106.

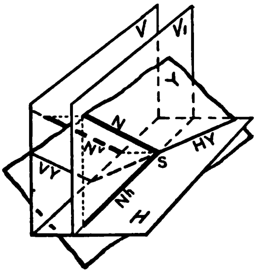


Fig. 107.

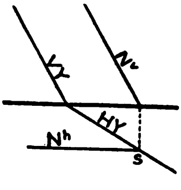


Fig. 108.

HX. Since *M* is parallel to *H*, its vertical projection (M^v, Fig. 106) must be parallel to *GL*. The vertical trace, *r*, of *M*, must lie in the vertical trace of the plane *X*.

Let the plane *Y*, Fig. 107, be intersected by the plane V_1, parallel to *V*, then the line of intersection *N* (a *V-parallel*) is parallel to the vertical trace of *Y*. As *N* is parallel to *VY*, the vertical projection of *N* (N^v, Fig. 108) must be parallel to *VY*. Since *N* is parallel to *V*, its horizontal projection (N^h, Fig. 108) must be parallel to *GL*. The horizontal trace, *s*, of *N* must lie in the horizontal trace of the plane *Y*.

32. **Lines of Maximum Inclination to H, and V.** These are the lines, of all which can be drawn in a plane, which make the *greatest* angle with *H* or *V*.

A line of maximum inclination to H or V is perpendicular to the corresponding trace of the plane containing the line. Let the line *A*, Fig. 109, be perpendicular to *HZ;* it is required to

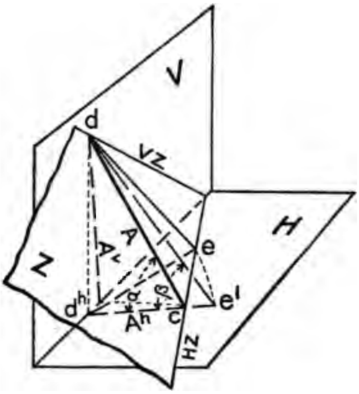


Fig. 109.

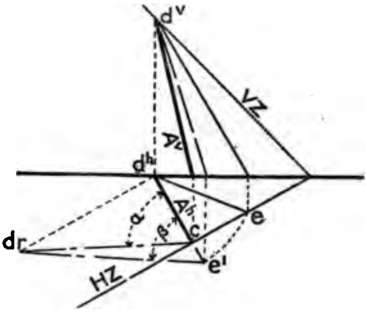


Fig. 110.

show that the angle between *H* and the line *A* is greater than the angle between *H* and any other line, not perpendicular to *HZ*, which can be drawn in the plane *Z*. Since line *A* is perpendicular to *HZ*, it is the shortest line that can be drawn from *HZ* to any point of line *A*, as, for instance, point *d* in which line *A* pierces *V;* that is, any line, *de,* drawn through *d* to any point *e* in *HZ* except *c* is longer than the line *A*. Hence, if *de* be revolved about the vertical line dd^h as axis until the horizontal projection of *de* coincides with that of *A*, the point *e* will fall outside of point *c*, as at *e'*. The angle $de'd^h$ will therefore be less than the angle dcd^h; that is, β is always less than α. Hence α is the greatest angle between *H* and any line in the plane *Z*. (See also Fig. 110.)

33. *The projections of a line perpendicular to a plane are perpendicular to the traces of the plane.* Let the line *K*, Fig. 111, be perpendicular to the plane *Q*. Since the line is perpendicular to the plane, it is perpendicular to any line which can be drawn in the plane. Draw an H-parallel, *M*, through the point in which line *K* rests on the plane *Q;* then *K* is perpendicular to *M*. As *M* is parallel to *H*, the horizontal projection of *K* must be perpendicular to the horizontal projection of *M* (Art. 30). But M^h is parallel to *HQ* (Art. 31), hence K^h is perpendicular to *HQ*. Similarly, K^v is perpendicular to the vertical projection of the V-parallel, *N*, drawn through the foot of *K;* hence K^v is perpendicular to *VQ*. (See also the same construction represented in projections, Fig. 112.)

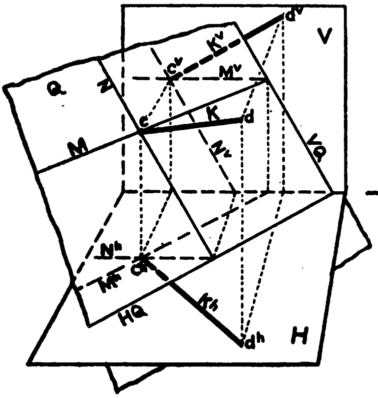


Fig. 111.

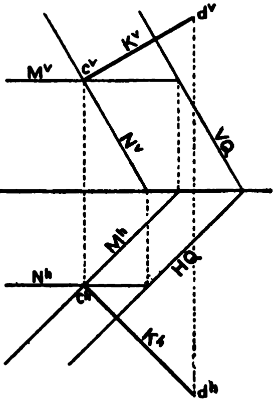


Fig. 112.

34. A plane can be passed through: (a) ***any three points, taken at random, but not more than three,*** (b) ***a point and a line,*** (c) ***two parallel lines,*** and (d) ***two intersecting lines.***

Problem 6. To pass a plane through two intersecting, or two parallel lines.

The solution of this problem depends upon the fact that the line of intersection with *V* or *H* of any plane which contains a line piercing *V* or *H* must contain the point in which the line pierces *V* or *H*. In other words, ***the traces of a line lying in a plane lie in the traces of the plane.*** If a line lying in a plane is doubly inclined, it has a vertical and a horizontal trace; if parallel to *V* or *H* and inclined to the other co-ordinate plane, it has but one trace; if parallel to *GL*, it can have neither vertical nor horizontal trace.

(*a*) Let the lines *A* and *B*, Fig. 113, intersect in the point *c*; it is required to pass a plane, *Q*, through the lines. Since *A* and *B* intersect, c^h and c^v must lie in the same ruled projector perpendicular to *GL* (Art. 29). The horizontal traces of *A* and *B* are s and s_1; through these points draw *HQ*, the horizontal trace of the required plane. The vertical traces of *A* and *B* are r and r_1; line *VQ*, containing r and r_1,

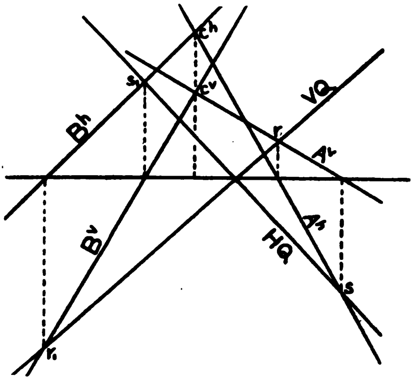


Fig. 113.

is the vertical trace of the required plane. The traces HQ and VQ must intersect GL in the same point. Visualize *the lines A and B, the plane Q, and the intersections of Q with H and V* (see Fig. 114).

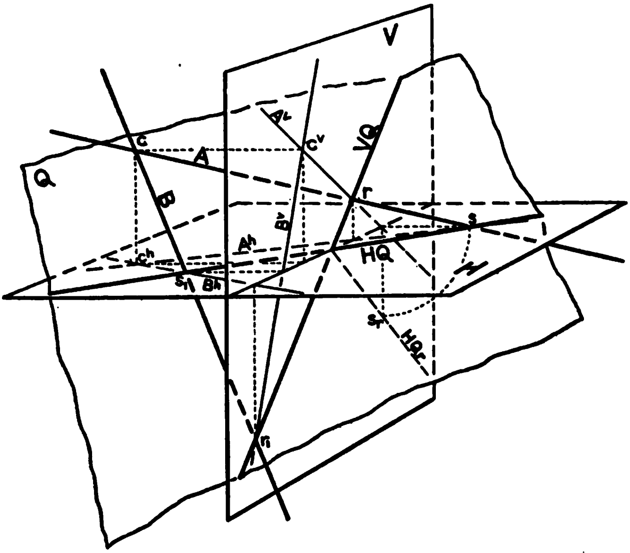


Fig. 114.

(*b*) *First Special Case.* Given the line M, Fig. 115, inclined to V and parallel to H, intersecting the line N inclined to H and parallel to V. Each of the lines can have but one trace, r and s, respectively. But M is an H-parallel, and N a V-parallel, of the required plane; hence HR, drawn through s, must be parallel to M^h, and VR, drawn through r, parallel to N^v. *Visualize M and N, the plane R, and its intersection with H and V.*

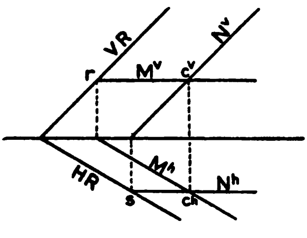


Fig. 115.

(*c*) Given the parallel lines *A* and *B*, Fig. 116, it is required to construct the plane *S* containing *A* and *B*. Find the traces of *A* and *B*; draw *HS* and *VS*. *Visualize A and B, the plane S, and the traces of S* (see Fig. 117).

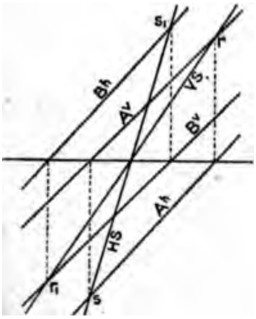


Fig. 116.

(*d*) *Second Special Case.* Given the lines *A* and *B* parallel to *GL*, Fig. 118. The traces of a plane passed through lines parallel to *GL* are parallel to *GL*. Find the profile traces, a^p and b^p, of *A* and *B* and through these traces draw the line a^pb^p produced to intersect *V* in *r*, *H* in s^p. Through *r*, the vertical trace of *ab*, draw *VT* parallel to *GL*. Counter-revolve s^p. Through *s*, the horizontal trace of *ab*, draw *HT* parallel to *GL*. *Visualize all conditions in Fig.* 118.

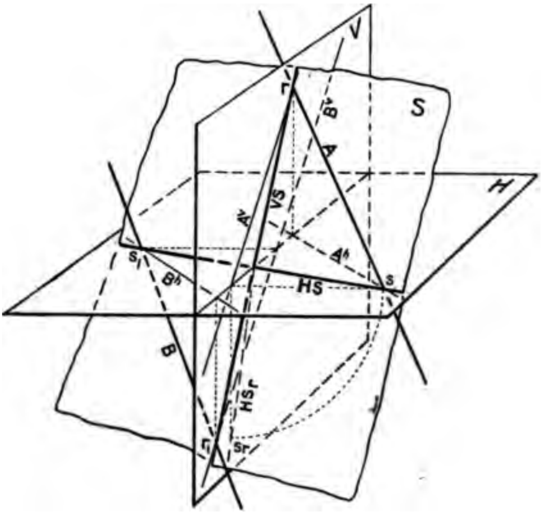


Fig. 117.

(*e*) If it were required to pass a plane, *U*, through a given line and a point not on the line (*A* and *c*, Fig. 119), draw through the given point *c* an auxiliary line, *bc*, intersecting the given line in any assumed point, *b*. Then pass the plane through *A* and *bc*. Instead of drawing the auxiliary line *bc* intersecting *A*, *bc* might be drawn parallel to *A* and the plane, *U*, passed through these lines. ***Visualize all conditions in Fig.*** 119.

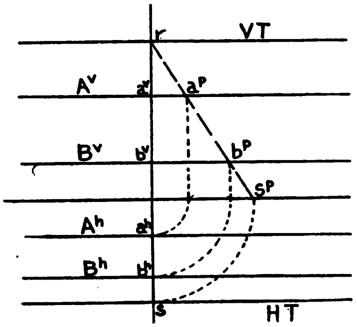


Fig. 118.

(*f*) If it were required to pass a plane, *W*, through three given points, Fig. 120, the points should be brought into two intersecting or two parallel lines and the plane passed through these lines. ***Visualize all conditions in Fig.*** 120.

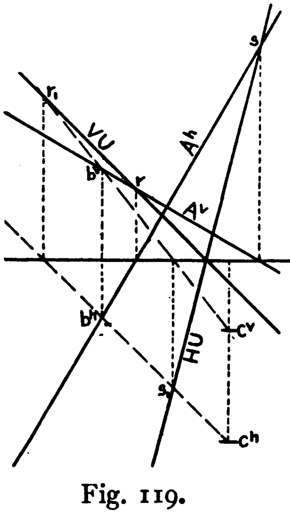


Fig. 119.

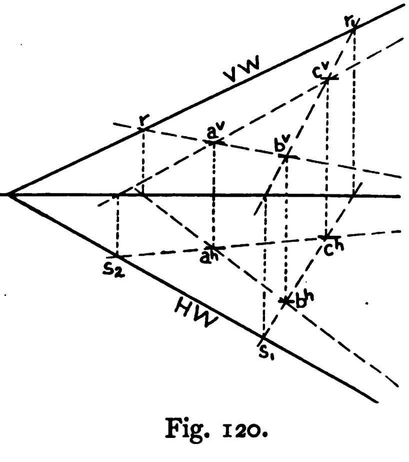


Fig. 120.

Problem 7. Given one projection of a point or a line lying in a plane, to find the other projection.

(*a*) Let a^h, Figs. 121 and 122, be the horizontal projection of a point lying in the plane *Q;* it is required to find a^v. In any convenient position, draw line *B* lying in *Q*, and containing the point *a*. Since the line lies in the plane, the traces of the line must lie in the traces of the plane; hence the intersection of B^h and *HQ* is the horizontal trace of *B*, s^h, vertically projected at s^v. The intersection of B^h and *GL*, r^h, is the horizontal projection of the vertical trace of *B*, vertically projected in *VQ* at r^v. Draw B^v, connecting r^v and s^v. Project from a^h to B^v, giving a^v, the required projection.

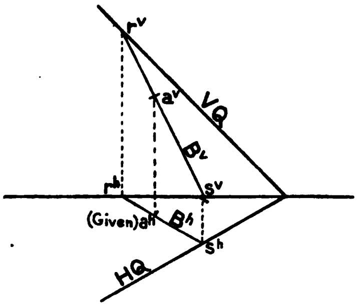


Fig. 121.

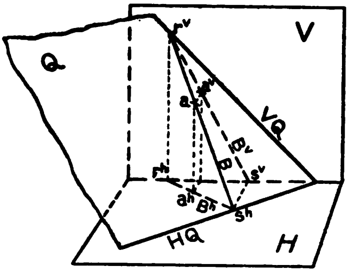


Fig. 122.

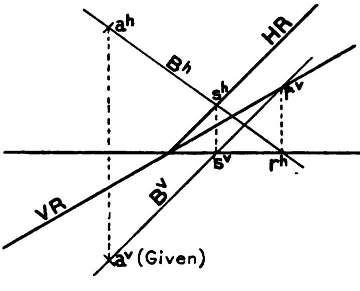


Fig. 123.

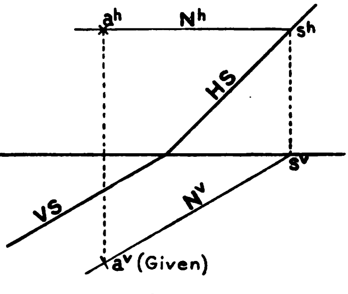


Fig. 124.

In a second example of the preceding method, Fig. 123, a^v is given. The line *B* lies in plane *R* and contains point *a*.

(*b*) In Fig. 124 the vertical projection of point *a*, lying in the plane *S*, is given, and a^h is found by means of the V-parallel, *N*, taken through *a*. *Visualize the plane S, point a lying in S, the V-parallel through a, and the projections of the V-parallel.*

(*c*) Given the vertical projection of a line *A*, Fig. 125, lying in the plane *T*. Produce *A* to intersect the traces of *T*. Find the horizontal projection of *A* produced, by means of the traces *r* and *s*, and then project the ends of the given line.

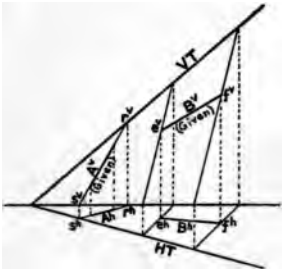


Fig. 125.

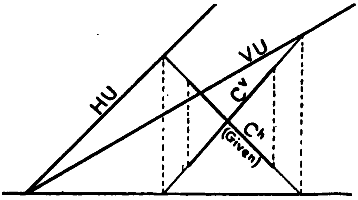


Fig. 126.

If the given projection of a line, as B^v, Fig. 125, falls nearly parallel to the trace of the plane, it may be best to find the required projection by means of the projections of the ends of the line, according to Fig. 121.

(*d*) *Additional Examples.* In Fig. 126 C^h is given, in Fig. 127 D^h is given, and in Fig. 128 E^h is given. Note the constructions, and decide by which of the preceding methods the *resultant* projection in each case is obtained. *Visualize each example.*

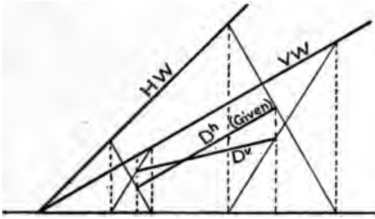


Fig. 127.

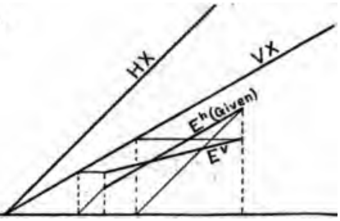


Fig. 128.

35. The Revolution of Planes.

A plane is determined when a point and a line in the plane are located.

(*a*) *First Case. When the axis does not lie in a co-ordinate plane.* Let Q, Fig. 129, be the given plane, and line A, perpendicular to H, the given axis. It is required to revolve Q through a given angle δ. Since it is perpendicular to H, the axis A pierces the plane Q in point o, horizontally projected in o^h coincident with A^h; the vertical projection of o is found by means of the H-parallel M (Prob. 7). Since point o lies in the axis of revolution A, it is fixed, and hence one point of the required plane. In its revolution about A, HQ moves in H perpendicular to A. Draw ce from the foot of A, and perpendicular to HQ. Revolve ce through the given angle δ, to ce_1; perpendicular to ce_1 draw HQ_1, one trace of Q revolved. To find the vertical trace, pass the H-parallel M_1 through the fixed point, o, and through the vertical trace r_1 of the parallel draw VQ_1.

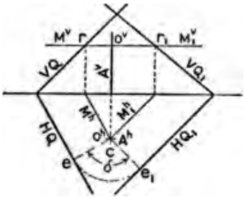


Fig. 129.

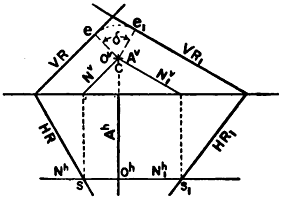


Fig. 130.

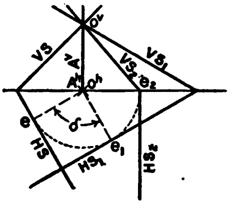


Fig. 131.

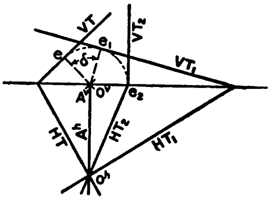


Fig. 132.

(*b*) Similarly, the plane *R*, Fig. 130, is revolved through the angle, δ, about axis *A* perpendicular to *V*.

(*c*) *Second Case. When the axis lies in a co-ordinate plane.* Let the axis *A*, Fig. 131, perpendicular to *H*, lie in *V*. Since *A* lies in *V*, the intersection, *o*, of *A* and the given plane, *S*, must be o^v, the intersection of A^v and *VS*. Revolved about *A*, *HS* falls at HS_1, one trace of *S* revolved, while VS_1 must pass through the fixed point, *o*, in *V*.

The plane *T*, Fig. 132, is revolved to the position T_1 (VT_1, HT_1) about the axis *A* perpendicular to *V* and lying in *H*.

(*d*). If the plane *S*, Fig. 131, be revolved until its trace, HS_2, is perpendicular to *GL*, then the plane is perpendicular to *V*. *The vertical trace, VS_2, is an edge view of the plane, and makes with GL the true angle between S and H* (Art. 25).

In Fig. 132, T_2 (VT_2, HT_2) shows *T* revolved until perpendicular to *H*.

Problem 8. To find the position of a point or a line lying in a plane, when the plane is revolved into H or V.

(*a*) First Case. *When the given plane is perpendicular to H or V.* Let the line *ab*, Fig. 133, lie in the plane *W*,

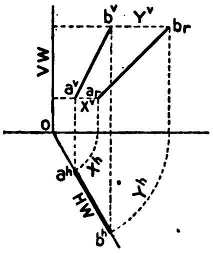


Fig. 133.

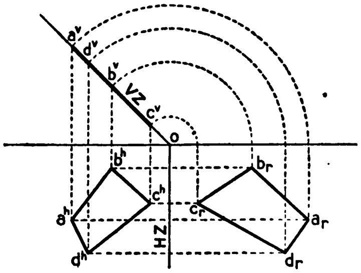


Fig. 134.

perpendicular to *H*. Revolve *W* about *VW* as axis into *V*. Since *VW* is perpendicular to *H*, the points *a* and *b* will move in the arcs of circles X^vX^h and Y^vY^h respectively, perpendicular to the axis of revolution, hence parallel to *H*.

(*b*) The plane *Z*, Fig. 134, containing the plane figure ***abcd***, is revolved about its horizontal trace into *H*.

(*c*) SECOND CASE. ***When the given plane is inclined to both H and V. First Method.*** Given, Fig. 135, the horizontal projection, a^h, of a point *a* lying in the plane *Q*; it is required to find the position of *a* when *Q* is revolved into *H*. Assume any line, *cd*, lying in *Q* and containing *a*. Draw the horizontal and vertical projections of *cd*, and find a^v (Problem 7).. In revolving *Q* into *H*, carrying with it the line *cd* and the point *a*, *HQ* is the axis of revolution. The end, *d*, of line *cd* is fixed, and points *a* and *c* move in planes perpendicular to *HQ*. Draw the traces, *HX* and *VX*, of the plane of revolution of point *c*, the horizontal trace perpendicular to *HQ*, the vertical trace perpendicular to *GL*. In order to show the projections of the path of point *c*, revolve *X* and *HQ* through 90° about *HX* as an axis, so that *X* coincides with *H* and *HQ* is seen endwise at point *e*. Then point *c* falls at c_r lying in VX_r perpendicular to *HX*. With *e* as centre, revolve c_r to c'_r, lying in *HX* (this revolution shows that point *c* has moved from *V* into *H*). Connect d^h with c'_r,

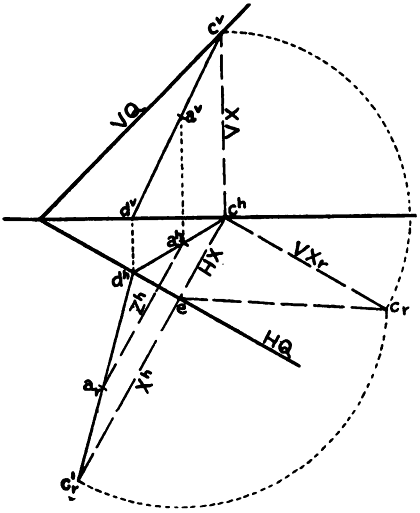


Fig. 135.

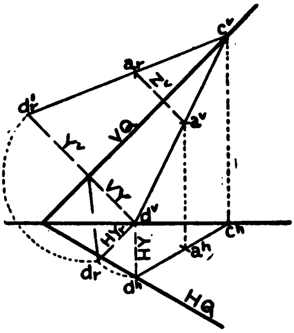


Fig. 136.

which gives *cd* revolved into *H*. The point *a*, revolving about *HQ*, moves in a plane *Z* parallel to *X*. Therefore draw Z^h intersecting $d^hc'_r$ in a_r, which is the required point.

In Fig. 136 the preceding point *a* is revolved into *V*. *Compare with Fig.* 135. *Visualize the planes of revolution Y and Z perpendicular to VQ; the plane Y and the line cd revolved into V; and the path of point a revolved into V.*

(*b*) *Second Method, by measurement with the dividers.* Let *a*, Fig. 138, be the given point in the plane *Q*. To revolve *a* into *H:* Pass a plane *X* perpendicular to *HQ* through point *a*. The plane *X* intersects *Q* in the line *ag*, which is the hypothenuse of the right triangle aa^hg. The base of the triangle is a^hg, its altitude aa^h, which equals a^vf. In Fig. 137, the base is found at a^hg^h, the altitude at a^vf. On *GL*, perpendicular to a^vf, lay off fg' equal to a^hg^h. Measure the hypothenuse q^vg', and lay it off from *HQ* on X^h, giving a^r, the required point. *Visualize Fig.* 137 *with the aid of Fig.* 138.

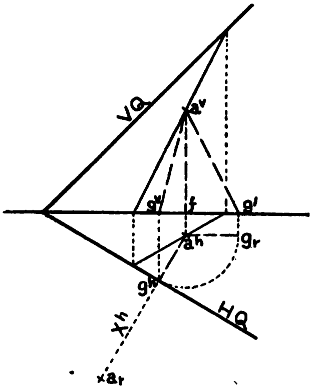


Fig. 137.

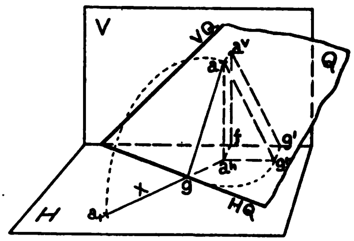


Fig. 138.

It will be noted that the triangle a^vfg' is the projection of triangle aa^hg revolved parallel to *V* (Figs. 137 and 138). In the practical use of this method it is not necessary to draw the triangle a^vfg'. Lay off fg' with the dividers, and then measure the distance a^vg'.

In Fig. 139 the preceding point, a, is revolved into V. *Visualize the triangle projecting $a^v g^v$ upon V, and find in the figure the measurement necessary to locate a_r.*

(*c*) In Fig. 140 the projections of a line ab, lying in the plane R, are given. The plane is revolved into H, and ab falls at $a_r b_r$. Observe that the line produced intersects HR in s. The position of a_r is found by the second method. Connecting a_r and s, the point b_r is found by projecting from b^h. If the trace s were not accessible, it would be necessary to revolve point b by the method used for point a.

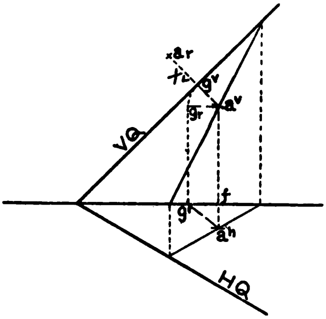


Fig. 139.

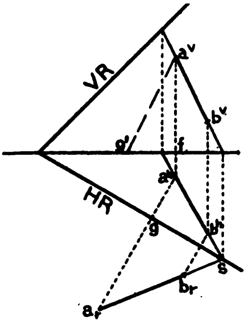


Fig. 140.

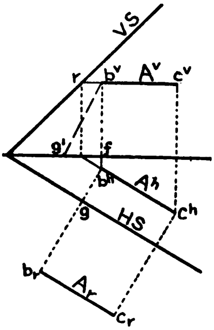


Fig. 141.

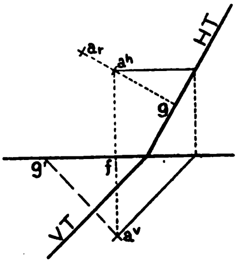


Fig. 142.

(*d*) The line A, lying in plane S, Fig. 141, is an H-parallel of the plane. Therefore, when S is revolved into H, A_r falls parallel to HS. Its distance from HS is equal to the perpendicular distance from *HS* to point *b*, found by the second method.

(*e*) The point *a*, Fig. 142, lies in the plane *T*. Revolving *T* into *H*, the point falls at a_r, found by the second method. *Visualize all conditions in this problem.*

36. *Intersecting Planes.* If, in any plane of projection, as *H*, *V*, or *P*, the traces of two planes intersect, then the planes must intersect. The intersection is a straight line common to the planes, and must pass through the intersection of the traces of the planes.

Problem 9. To find the intersection of two planes.

Case A. When both pairs of traces intersect. Let *Q* and *R*, Fig. 143, be the given planes. The points *r* and *s* are the traces of the required intersection *A*; A^v and A^h are found according to Problem 5.

In Fig. 144 it is first necessary to find the point *s* by producing *HS* and *HT*; A^h and A^v are then determined as in the preceding example. *Visualize the planes and the intersection A.*

Case B. An oblique plane intersected by a plane parallel to a co-ordinate plane. Let the plane *U*, Fig. 145, be intersected by the plane *W*, parallel to *H*. The intersection, *A*, must pass through the intersection, *r*, of the vertical traces of the planes; and, since *W* is parallel to *H*, *A* must be an H-parallel of the plane *U* (Art. 31 and Fig. 106).

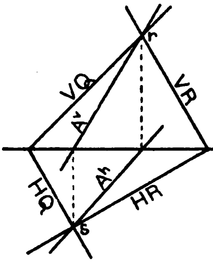


Fig. 143.

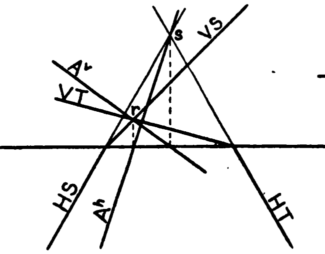


Fig. 144.

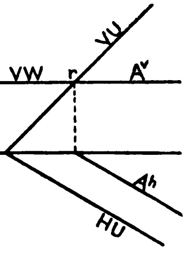


Fig. 145.

Case C. When one pair of traces intersect and the other pair are parallel. Let *VQ* and *VR*, Fig. 146, intersect, and *HQ* and *HR* be parallel. Since the horizontal traces of the planes are parallel, the point of intersection of the traces is infinitely distant. Hence, the line of intersection of the planes, passing through this infinitely distant point, must be an H-parallel common to the planes, and passing through the intersection, *r*, of their vertical traces.

Case D. When one or both pairs of traces do not intersect within the limits of the drawing. In this case the intersection of the planes must be determined by means of auxiliary planes, which are usually taken parallel to *H* or *V*. In Fig. 147 the intersection of *VS* and *VR* is regarded as inaccessible. The auxiliary plane, *X*, parallel to *V*, is taken in any convenient position. This plane cuts the planes *R* and *S* in the V-parallels *N* and N_1, respectively, which must intersect in the intersection of the planes. The line *A*, drawn through the points *s* and *d*, is the required intersection.

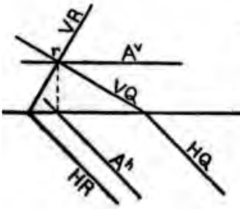


Fig. 146.

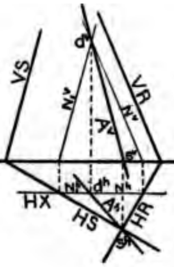


Fig. 147.

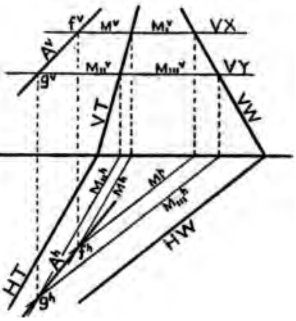


Fig. 148.

When both pairs of traces, Fig. 148, do not intersect within the limits of the drawing, two auxiliary planes must be used. The planes *X* and *Y*, parallel to *H*, cut from the planes *T* and *W* the H-parallels *M*, M_1, M_{11}, and M_{111}, respectively. The lines *M* and M_1 intersect in the point *f*, the lines M_{11} and M_{111} in point *g*. The line *A*, containing *f* and *g*, is the required intersection.

Case E. Both planes parallel to GL. The intersection is parallel to *GL*. Find the profile traces of the planes, as *PQ* and *PR*, Fig. 149. Counter-revolve the intersection, t^p, of the traces, and through the horizontal and vertical projections of *t* draw the projections of *A*, the required intersection.

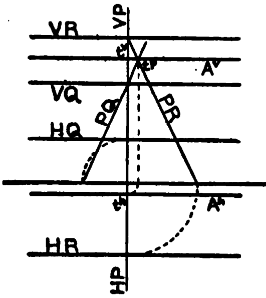


Fig. 149.

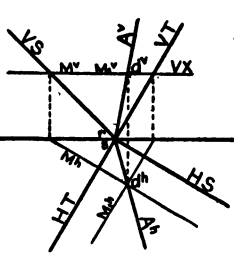


Fig. 150.

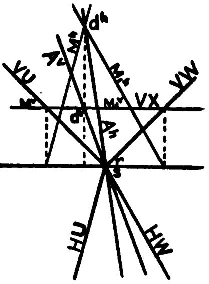


Fig. 151.

Case F. When the intersection of both the vertical and horizontal traces fall in GL. Two points are necessary to determine the line of intersection. In this case, as the intersections, *r* and *s*, of the respective traces coincide, it is necessary to determine a second point in the line of intersection of the planes by means of an auxiliary plane, usually taken parallel to *H* or *V*. Let the traces of the planes *S* and *T*, Fig. 150, intersect in *GL*. The auxiliary plane *X*, parallel to *H*, cuts from the planes *S* and *T* the H-parallels *M* and M_1, intersecting in *d*. The line *A*, passing through *d*, *r*, and *s*, is the required intersection. *Visualize the planes T, S, and X, the H-parallels, and the line A.*

The same method is used in the case of the planes, *U* and *W*, Fig. 151, intersecting in *GL*. *Visualize all conditions.*

Case G. An oblique plane intersected by a plane passing through GL. Let the plane *R*, Fig. 152, be intersected by the plane *Q*, which passes through *GL* from the first to the third quadrants, making 30° with *H*. Find the profile traces of *Q* and *R;* counter-revolve the intersection, t^p, of these traces, and through the horizontal and vertical projections of *t* draw the projections of *A*, the required intersection.

A second example of the same method is shown in Fig. 153. *Visualize all conditions.*

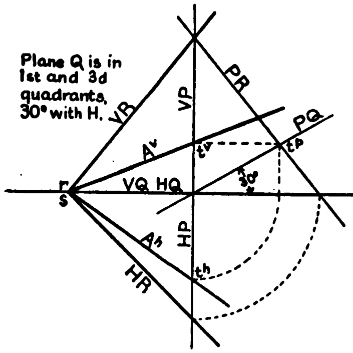


Fig. 152.

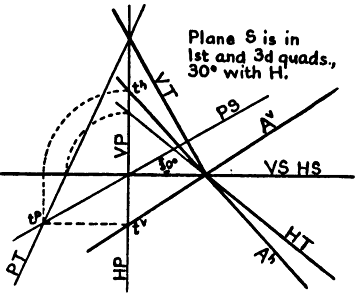


Fig. 153.

37. *The intersection of a line and a plane.* (*a*) If a line which is neither contained in nor parallel to a plane be produced, it must intersect the plane. Any plane passed through a line which pierces a given plane must intersect the given plane; and the point of intersection of the line and the given plane must be in the line of intersection of the two planes.

(*b*) An infinite number of planes can be passed through any line. If the traces of any plane be drawn through the traces of a line, the plane must contain the line; thus, for example, in Fig. 154, the planes *Q*, *R*, *S*, and *T* contain the line *A*.

Problem 10. To find the intersection of a line and a plane.

(*a*) *General Method.* Let it be required to find where the line *A*, Figs. 155 and 156, pierces the plane *Q*. Find the traces *r* and *s*, of *A*, Fig. 155. Pass any plane, as *X*, through *A*, first assuming the trace *VX* through *r*, and then drawing the trace *HX* through *s* and the point where *VX* cuts *GL* (*b*, Art. 37). Find the line of intersection, *B*, of the planes *Q* and *X*, Since the line *B*, common to *Q* and *X*, lies in the plane containing line A, the intersection of A and B is the point in which A pierces the plane Q. The projections A^v and B^v produced intersect in c^v, the vertical projection of the required point.

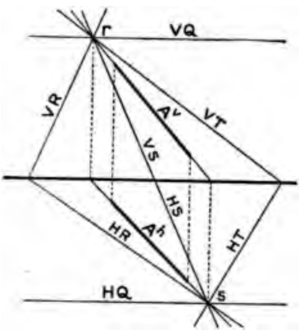


Fig. 154.

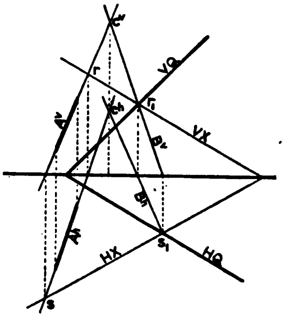


Fig. 155.

The projections A^h and B^h produced intersect in c^h, the horizontal projection of the required point. Test the construction

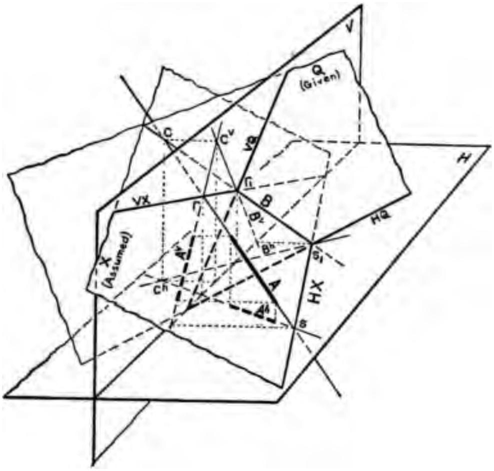


Fig. 156.

by noting if c^v and c^h lie in the same ruled projector perpendicular to GL. It will be seen that A pierces Q in the second quadrant. *Visualize each step in the construction.*

(*b*) A second example, Fig. 157. The explanation and lettering are the same as in example (*a*).

(*c*) A third example, Fig. 158. The explanation and lettering are the same as in example (*a*).

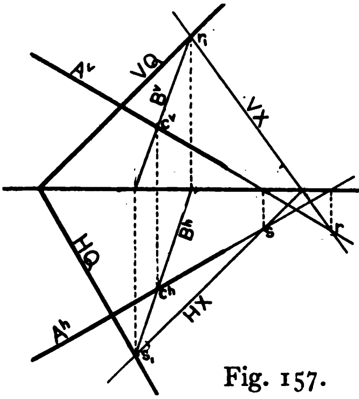


Fig. 157.

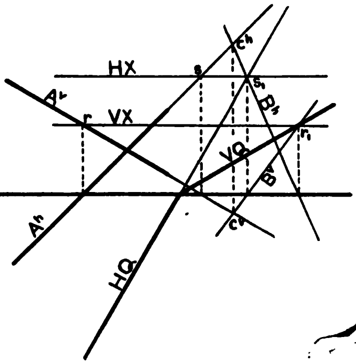


Fig. 158.

(*d*) *The Usual Method.* Ordinarily, the most direct method of finding the intersection of a line and a plane is to pass through the line a plane perpendicular to *H* or *V*. Let *S*, Fig. 159, be the given plane, *A* the given line. Pass through *A*

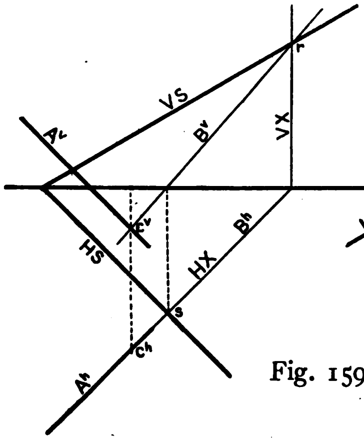


Fig. 159.

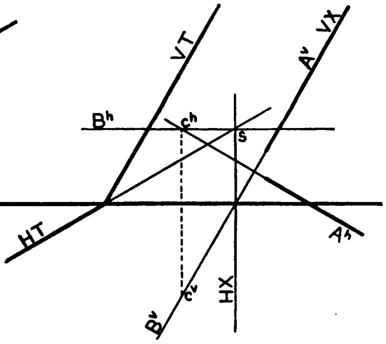


Fig. 160.

the plane *X* perpendicular to *H*. The intersection of the planes *S* and *X* is the line *B*. The vertical projections of the lines A and B intersect in c^v, the vertical projection of the point in which line A pierces the plane S. The horizontal projection of point c lies in B^h which coincides with A^h.

In Fig. 160 the intersection of line A with the plane T is obtained by passing through A a plane perpendicular to V. It should be noted that, as A^v is parallel to VT, VX, coinciding with A^v, must be parallel to VT. Hence the line of intersection, B, of the planes T and X, is parallel to V (Case C, Prob. 9).

(e) *Special Case. To find the intersection of a line and a plane determined by two lines, intersecting or parallel.* It is required to find the point in which the line A, Fig. 161, pierces

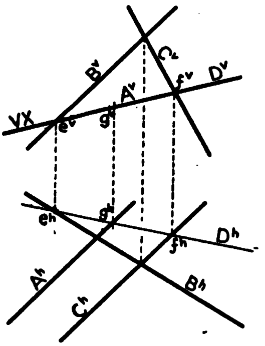


Fig. 161.

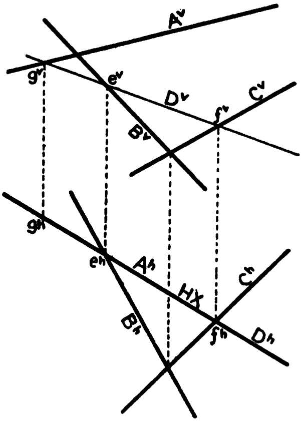


Fig. 162.

the plane (BC) determined by the intersecting lines B and C. Pass a plane X, perpendicular to V, through line A. The plane X cuts B^v in e^v, and C^v in f^v. Line D^v, connecting e^v and f^v, is the vertical projection of D, the intersection of the planes X and BC. Find D^h. The point, g, in which A cuts D, is the required intersection. The plane passed through A may be perpendicular to V or H according to which plane gives the clearer construction.

In the second example, Fig. 162, the plane X is taken perpendicular to H.

In the third example, Fig. 163, in which the plane is determined by two parallel lines in the third quadrant, the plane *X* is taken perpendicular to *V*.

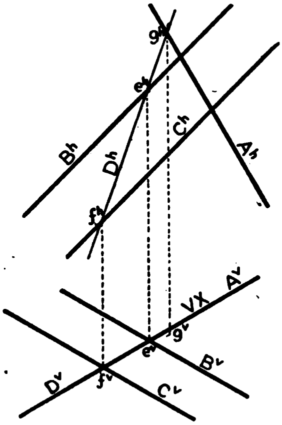


Fig. 163.

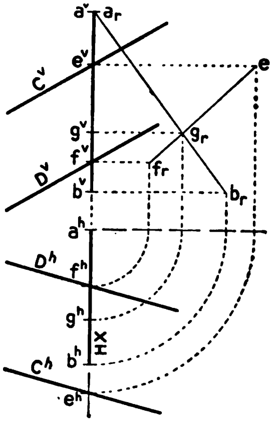


Fig. 164.

In the fourth example, Fig. 164, the given line *ab* lies in a profile plane. Passing through *ab* the plane *X* perpendicular to *H*, the given lines *C* and *D* intersect *X* in points *e* and *f*. The required point, *g*, is the intersection of the profile lines *ab* and *ef*, here found by revolving the lines to a position parallel to *V*, about an axis perpendicular to *H* and passing through point *a*.

(*f*) *To find the intersection of two planes, when each is determined by two lines, intersecting or parallel* (*application of the preceding case*). Let it be required to find the intersection of the limited plane surfaces *AB* and *CD*, Fig 165. The intersection is a straight line connecting the two points in which two lines of either plane pierce the other plane. To find where the line *A*, of plane *AB*, pierces the plane *CD*, pass through *A* a plane, *X*, perpendicular to *H*. The plane *X* intersects plane *CD* in line *ef*. The line *A* pierces *ef* in point *g*. Pass through the line *B* a plane, *Y*, perpendicular to *H*. The plane *Y* cuts the plane *CD* in the line *jk*. It will be seen that,

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in order to find the point, *m*, where *B* cuts *jk*, it is necessary to produce the latter; that is, to produce the plane *CD*. Connecting the projections of points *g* and *m* gives line *gn*, the required intersection.

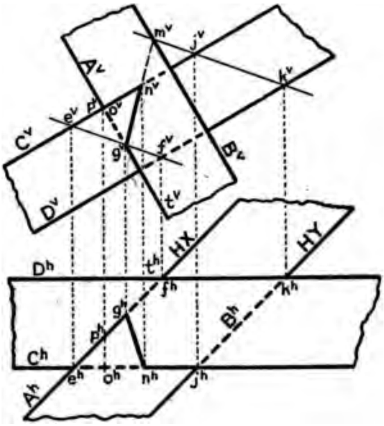

Fig. 165.

38. Visibility. This term signifies whether, in the case of planes and solids regarded as opaque, particular points, lines, and surfaces are visible or hidden, the same being determined by inspection or otherwise.

Let the two planes, Fig. 165, be considered opaque, and let it be supposed that there is doubt as to which of the lines o^vn^v and p^vg^v is visible. Since the lines A and C in space do not intersect (Art. 29), it is evident that the point in which the projections A^v and C^v intersect represents two points, p and o, one in A, the other in C. The visibility of o^vn^v or p^vg^v depends upon whether p or o is nearer the eye; that is, which point is *in front* of the other. Projecting from the intersection of A^v and C^v to A^h and C^h, respectively, it is evident that o is in front of p; pn is therefore visible. Again, if it were required to prove whether g^hf^h is visible or invisible, project from the intersection of A^h and D^h, to A^v and D^v. It is evident that the point t is below f; g^hf^h is therefore *invisible*.

Problem 11. To find the projections and true length of the shortest distance from a point to a plane.

This distance must lie in a line containing the point and perpendicular to the plane. Let A, Fig. 166, be the given point, and Q the given plane. The projections B^v and B^h, drawn perpendicular to the traces of the plane, represent a line in space, B, perpendicular to the plane Q (Art. 33). The line B contains point a, since the respective projections of the line contain a projection of the point. Line B meets plane Q at point c (Prob. 10); a^vc^v and a^hc^h are the projections of the required distance. To find the actual distance, revolve ac into V or H (Prob. 1).

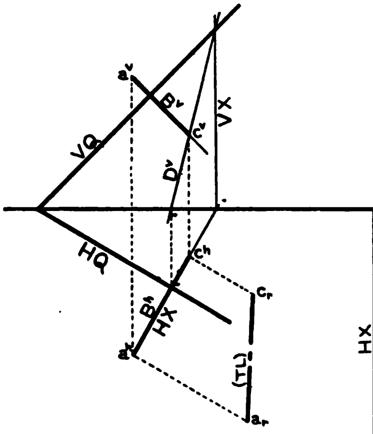

Fig. 166.

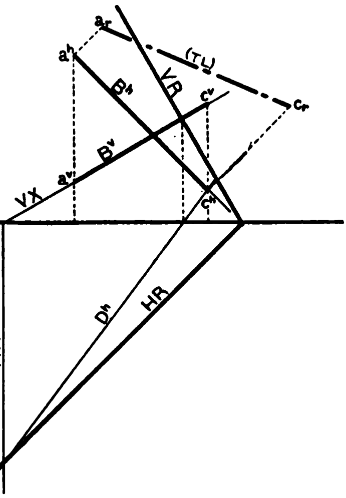

Fig. 167.

Additional examples are given in Figs. 167 and 168. *Visualize each construction.*

Problem 12. To project a line upon an inclined plane.

It is required to project the line A, Fig. 169, upon the plane Q. The space projectors, B and C, of the ends of the given line must be perpendicular to the given plane. Hence draw the projections of *B* and *C* perpendicular to the traces of *Q* (Art. 33), and find the points *d* and *e* where *B* and *C* pierce *Q* (Prob. 10). The line *F*, connecting *d* and *e*, is the required projection of the line *A* on the plane *Q*.

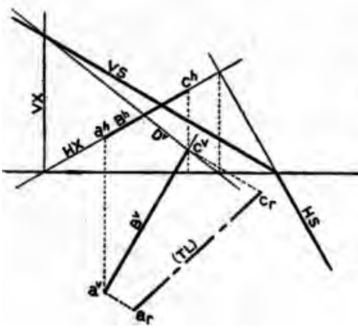

Fig. 168.

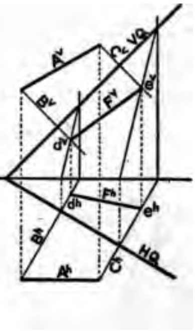

Fig. 169.

Problem 13. Through a given point, to pass a plane which shall be parallel to a given plane.

(*a*) Let *a*, Figs. 170 and 171, be the given point, *Q* the given plane, and *R* the required plane. The solution of this problem depends upon the use of an H- or a V-parallel. Since the respective traces of the given and required planes must be parallel, and the latter contain the point *a*, it is evident that the line

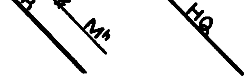

Fig. 170.

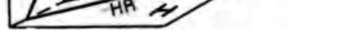

Fig. 171.

M, Fig. 171, parallel to *HQ*, and drawn through point *a*, is an H-parallel of the required plane. Through the projections of point *a*, Fig. 170, draw the projections of the H-parallel *M*. Through the vertical trace, *r*, of *M*, draw *VR* parallel to *VQ*. Through the intersection of *VR* and *GL* draw *HR* parallel to *HQ*.

(*b*) In the second example, Fig. 172, the required plane *T* is found by means of the V-parallel, *N*, passed through the given point *a*. (See also Fig. 173.)

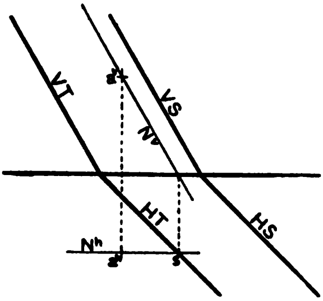

Fig. 172.

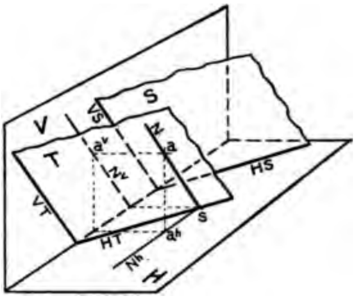

Fig. 173.

(*c*) *Special Case.* Let the given plane *Q*, Fig. 174, be parallel to *GL*. In this case the above method fails, but the required plane, *R*, can be passed through the given point *a* by means of a profile trace of *R*. Through the given point, *a*, pass the profile plane *P*. Find the profile projection, a^p, of point *a*, and the profile trace, *PQ*, of plane *Q*. Through a^p draw the profile trace, *PR*, of the required plane parallel to *PQ*. From the profile trace, *PR*, find the vertical trace, *VR*, and the horizontal trace, *HR*, of the required plane. *Visualize the construction.*

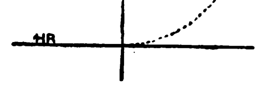

Fig. 174.

Problem 14. Through a given point, to pass a plane which shall be perpendicular to a given line.

(*a*) The solution of this problem depends upon the use of an H- or V-parallel, and the fact that, if a plane is perpendicular to a line, the traces of the plane are perpendicular to the projections of the line (converse of Art. 33). Let *c*, Fig. 175, be the given point, and *A* the given line. Since the horizontal trace of the required plane, *S*, is perpendicular to A^h, the horizontal projection of any H-parallel taken in *S* must be perpendicular to A^h. Through c^h draw M^h perpendicular to A^h, and through c^v draw M^v parallel to *GL;* then line *M* is an H-parallel of the required plane. Find the vertical trace, *r*, of *M*, and through *r* draw *VS* perpendicular to A^v. Through the intersection of *VS* and *GL* draw *HS* perpendicular to A^h. The plane *S* contains point *c*, and is perpendicular to line *A*.

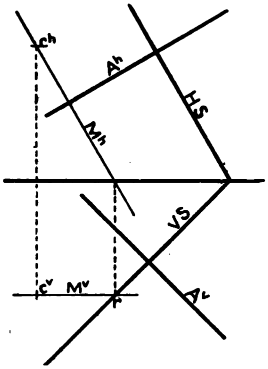

Fig. 175.

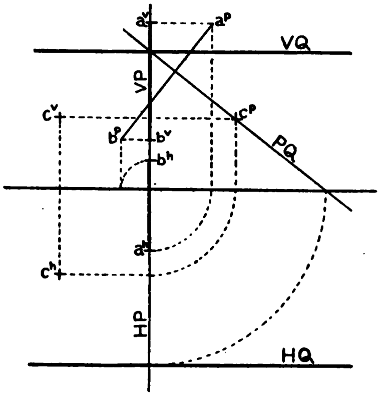

Fig. 176.

(*b*) *Special Case.* Let *ab*, Fig. 176, be the given line, and *c* the given point. In this case the above method fails. Since the projections of the given line are perpendicular to *GL*, the traces of the required plane *Q* will be parallel to *GL*, hence, in order to draw *Q* perpendicular to *A*, a profile plane may be used. Pass the auxiliary profile plane *P* through the line *ab*. Find the profile projection, a^pb^p, of *ab*, and the profile projection, c^p, of point c. Through c^p draw the profile trace, PQ, of the required plane perpendicular to a^pb^p. From the profile trace, PQ, find the vertical trace, VQ, and the horizontal trace, HQ, of the required plane. *Visualize the construction.*

Problem 15. Through a given line, to pass a plane which shall be parallel to another given line.

Let it be required to pass through the line A, Fig. 177, a plane parallel to the line B. The solution of this problem depends upon the fact that any plane passed through one of two parallel lines is parallel to the other. Through any point of A, as c, pass the line D parallel to line B; that is, draw D^h parallel to B^h, and D^v parallel to B^v (Art. 28). Find the plane, R, containing the intersecting lines A and D (Problem 6). Then R will be parallel to B, since it contains line D, parallel to B; hence R is the required plane.

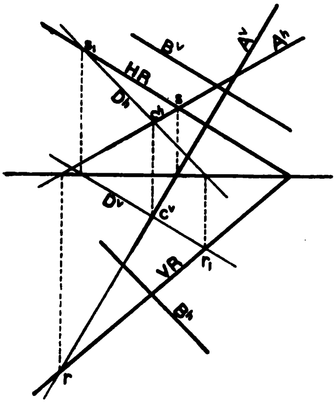

Fig. 177.

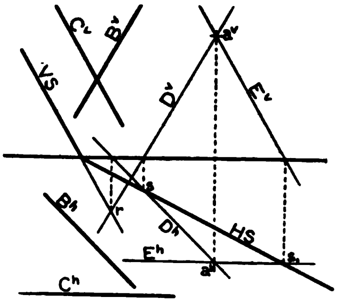

Fig. 178.

Problem 16. Through a given point, to pass a plane which shall be parallel to each of two given lines.

Let it be required to pass through the point a, Fig. 178, a plane parallel to the lines B and C. Through point a draw line *D* parallel to *B*, and line *E* parallel to line *c* (Art. 28). Pass the plane *S* through the intersecting lines *D* and *E* (Problem 6). Then *S* is the required plane, since it contains a line parallel to each of the given lines.

Problem 17. Through a given line, to pass a plane which shall be perpendicular to a given plane.

(*a*) Let it be required to pass through the line *A*, Fig. 179, a plane which shall be perpendicular to the plane *Q*. The solution of this problem depends upon the fact that, if a line is perpendicular to a given plane, any plane passed through the line will be perpendicular to the given plane. From any point of line *A*, as *c*, drop a perpendicular, *B*, to plane *Q*; that is, draw B^v perpendicular to *VQ*, and B^h perpendicular to *HQ* (Art. 33). Pass the plane *R* through the intersecting lines *A* and *B* (Problem 6). Then *R* is the required plane; it contains line *A*, and is perpendicular to plane *Q*, since it contains line *B*, perpendicular to *Q*.

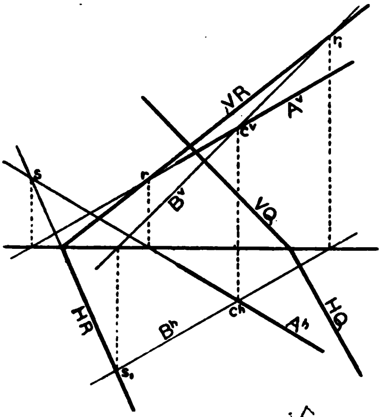

Fig. 179.

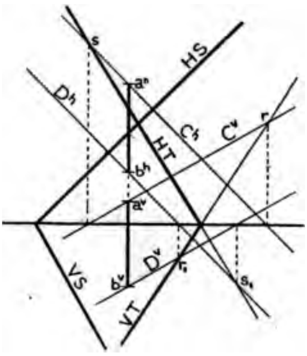

Fig. 180.

(*b*) *First Special Case.* Let it be required to pass through the profile line *ab*, Fig. 180, a plane perpendicular to the plane *S*. Instead of the preceding method, a simpler one is to proceed as follows: Through the points *a* and *b* draw the lines *C* and *D*, respectively, perpendicular to the plane *S*. Pass the plane *T* through the parallel lines *C* and *D* (Problem 6); then *T* is the required plane.

(*c*) *Second Special Case.* Let it be required to pass through the line *A*, Fig. 181, a plane perpendicular to the plane *U*

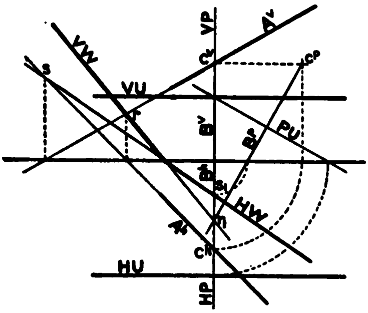

Fig. 181.

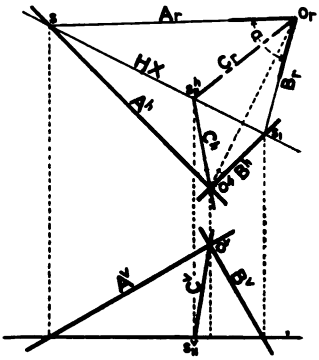

Fig. 182.

parallel to *GL*. From any point in line *A*, as *c*, draw the line *B* perpendicular to plane *U*. It is obvious that the line *B* is a profile line, and that it is necessary to use a profile projection to find its traces. Pass the auxiliary profile plane, *P*, through line *B*, and find the profile projection, c^p, of the point *c*. Draw the profile trace, *PU*, of the plane *U*. The profile projection of *B*, B^p, passes through c^p, and is perpendicular to *PU*. Find the traces, r_1, and s_1, of *B*; also the traces, *r* and *s*, of *A*. Through these traces draw the traces of *W*, the required plane.

Problem 18. To find the angle between two intersecting lines and the projections of the bisector of the angle.

(*a*) Let *A* and *B*, Fig. 182, be the given lines intersecting in point *o*. The plane of angular measurement, *X*, of two intersecting lines is the plane which contains both lines. If one trace of the plane of measurement be found, and the plane, together with the given lines, be revolved about this trace into a co-ordinate plane, the true size of the angle will be shown. Find the horizontal trace, *HX*, of the plane containing lines *A* and *B* (Problem 6). Revolve *X* about *HX* into *H* In this revolution the points *s* and s_1 remain fixed, while point *o* moves in a plane perpendicular to *HX*. Find the revolved position of point *o* according to the Second Method, Problem 8. Draw *A* and *B* revolved, giving α, the required angle.

(*b*) *To draw the projections of the bisector of the angle.* Let it be required to bisect the acute angle. Draw the bisector, C_r, of the angle α. In order to show the vertical and horizontal projections of the line *C*, it is necessary to counter-revolve this line into the plane *X*. Produce C_r to intersect *HX* in $s_{11}{}^h$. In the counter-revolution, the point s_{11} remains fixed, while the point *o* moves in a plane perpendicular to *HX* until it reaches its original position *o* (o^h,o^v). Find the vertical projection of the point s_{11} and draw C^h and C^v, the required projections of the bisector.

(*c*) In the second example, Fig. 183, the bisector of the acute angle α is the line C_r. It will be seen that, as C_r does not intersect *HX* on the paper, the preceding method of counter-revolution fails. The method used consists in counter-revolving an H-parallel, M_r, passed through any assumed point, p_r, in the bisector, and intersecting, as at e_r, the given line B_r. Having drawn M_r parallel to *HX*, counter-revolve the point *e*, giving e^h and e^v, and through these points draw the projections, M^h and M^v, of the H-parallel. Counter-revolve the point *p*, giving p^h and p^v. Through the projections of points *o* and *p* draw C^h and C^v, the required projections of the bisector.

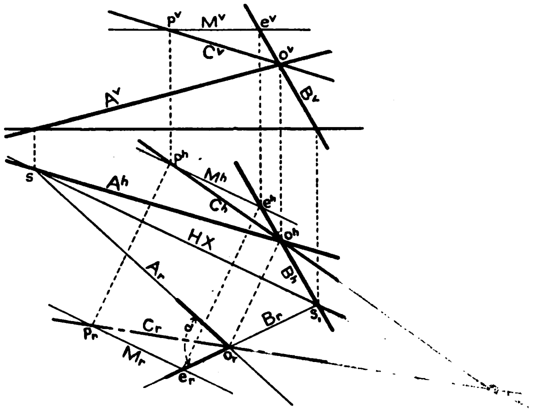

Fig. 183.

Problem 19. To find the angle between a line and an inclined plane.

(*a*) Let it be required to find the angle α between the line *A* and the plane *Q*, Fig. 184. If from any point, as *c*, in the line *A*, a perpendicular, *B*, be dropped to the plane *Q*, the perpendicular must pass through the projection of *A* on *Q*. The given line *A*, the perpendicular *B*, and the projection of *A* on *Q* together form a right triangle, of which the projection of *A* on *Q* is the *base*, the perpendicular *B* the *altitude*, and the given line *A* the *hypothenuse.* The angle between line *A* and its projection on the plane *Q* is the angle which *A* makes with *Q*. But this angle is the complement of the angle between *A* and the perpendicular *B;* hence, if the latter angle be found independently of the projection of *A* on *Q*, it may be used as the vertex of a right triangle in which the required angle can be measured. Through any point, as *c*, in the line *A*, Fig. 184, pass a line, *B*, perpendicular to the plane *Q* (Art. 33). Find the angle, β, between the lines *A* and *B* (Problem 18); this angle, between c_rs and c_rs_1, is the complement of the required angle. In any convenient position draw a line perpendicular to one of these lines, as c_rs_1, thus forming a right triangle, and giving α, the required angle between the line *A* and the plane *Q*.

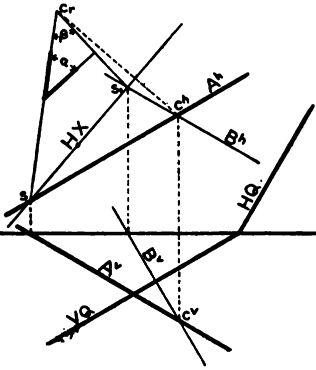

Fig. 184.

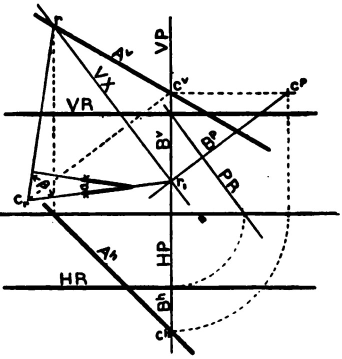

Fig. 185.

(*b*) In the second example, Fig. 185, the given plane *R* is parallel to *GL*. Hence the perpendicular *B*, passed through any point *c* in *A*, must be projected upon *H* and *V* perpendicular to *GL*. Find the vertical trace, r_1, of *B*, by means of the profile plane *P* (compare with *c*, Problem 17). Point *r* is the vertical trace of *A*, and the line *VX*, drawn through *r* and r_1 is the vertical trace of the plane containing lines *A* and *B*. Revolving *A* and *B* into *V*, the angle between them is β. The required angle, α, the complement of β, is obtained by constructing a right triangle as shown.

CHAPTER VI.

THE DOUBLE GROUND LINE; AUXILIARY PLANES OF PROJECTION; PROBLEMS ON THE LINE AND PLANE CONCLUDED.

Problem 20. Given one projection and three points in the other projection of a plane figure, to find its true shape and size, and to find the projections of the bisector of one angle of the figure.

(*a*) *To complete the projections.* One projection of any plane figure can be assumed; also, a plane can be passed through any three points taken at random, but not more than three. Let $a^h b^h c^h d^h$, Fig. 186, be the given projection, and a^v, b^v, and c^v the given points of the other projection. Pass a plane, *X*

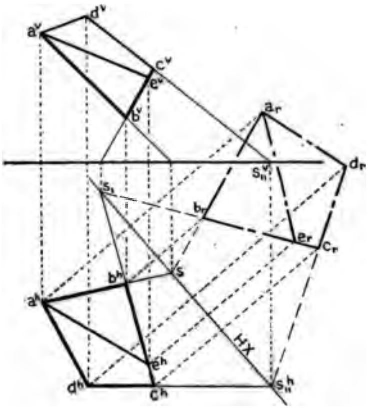

Fig. 186.

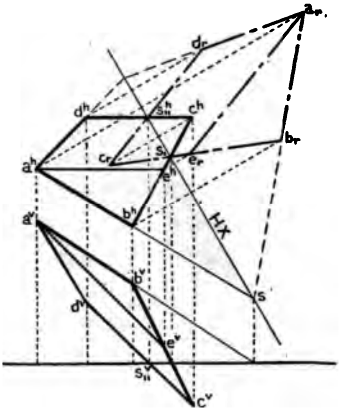

Fig. 187.

through the three points *a*, *b*, and *c* (Problem 6); the horizontal trace, *HX*, is sufficient. The point *d* must lie in the plane *X;*

it therefore is necessary to locate d^v so as to satisfy this condition. Through d^h draw the horizontal projection of any line contained in X; that is, a line passing through any point of the complete projection and intersecting HX. In this case, for convenience, the line d^hc^h, intersecting HX at $s_{11}{}^h$, is taken. Find the vertical projection $c^vs_{11}{}^v$, and locate d_v by projecting from d^h (compare with Problem 7).

(*b*) *To find the true shape and size of the plane figure.* Revolve the points a, b, c, and d into H (Problem 8, Second Method), and connect the revolved positions of the points. Check the construction by noting whether the horizontal projections of the various lines of the figure, and their corresponding revolved positions, produced if necessary, intersect on HX.

(*c*) *To draw the projections of the bisector of the angle bad.* Draw the bisector a_re_r, intersecting b_rc_r at e_r. Counter-revolve the points a and e, and draw a^he^h and a^ve^v, the required projections.

(*d*) In the second example, Fig. 187, $a^hb^hc^hd^h$ is the given projection, and a^v, b^v, and c^v the given points of the other projection. The lettering and explanation are the same as for Fig. 186. It should be noted that, in this case, the axis of revolution, HX, intersects the horizontal projection of the figure, since the points a, b, and d are above H, while the point c is below H. After revolution, the points a, b, and d will be found on one side of HX, while the point c will be found on the opposite side.

39. The Double Ground Line. There are several reasons for the use of the double ground line (Art. 12). Some persons can visualize a drawing more easily when the edge views of H and V are thus separated than when both of these views are represented by one line. Again, if the views of an object are so large that each view takes a separate sheet of paper, it will be seen that, if the construction requires a ground line, the latter must be repeated on each sheet, and becomes a double ground line. Finally, there are cases where the assumption of a second ground line will bring the construction into an available space, or otherwise simplify it; compare Figs. 188 and 189, in which Prob. 20 is solved, the same data being used in each case.

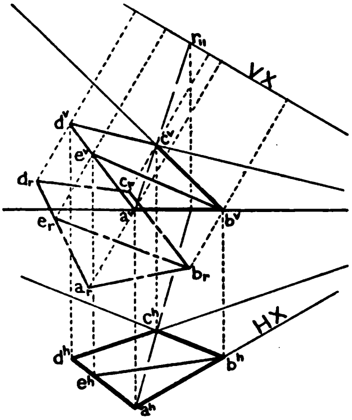

Fig. 188.

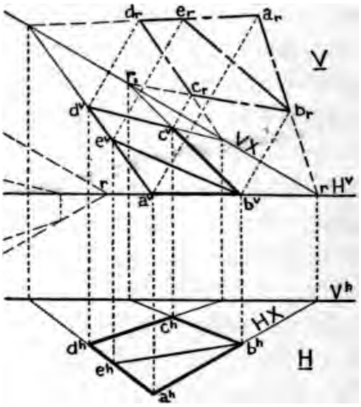

Fig. 189.

40. Auxiliary Planes of Projection. It will be seen that the method of Fig. 189, as compared with Fig. 188, is equivalent to assuming an auxiliary vertical plane of projection, parallel to and in front of the original vertical co-ordinate plane. In many cases a solution can be simplified by the use of an auxiliary vertical plane of projection *not parallel to the original one,* but placed in some advantageous position with respect to the data.

The general idea of such an auxiliary plane of projection is illustrated in Figs. 190 and 191. Let it be supposed that the solution of a problem requires a construction with respect to the true length of a given line, as *ab*, Fig. 190. If the auxiliary vertical plane, V_1, be placed parallel to *ab*, the true length of *ab* will be projected on V_1. Since the line projectors aa^v and aa_1^v of the point in space, *a*, are each parallel to *H*, the distance $a_1^vc_1$ is equal to a^vc; that is, the projection a_1^v is the same distance from G_1L_1 that a^v is from *GL*. Similarly, $b_1^vd_1$ is equal to b^vd. The construction is as follows: Place the auxiliary ground line, G_1L_1, Fig. 191, parallel to and at any convenient distance from a^hb^h. Draw from a^h and b^h the ruled projectors perpendicular to G_1L_1. Make $c_1a_1^v$ equal to ca^v, and $d_1b_1^v$ equal to db^v. Draw $a_1^vb_1^v$, the required projection on the auxiliary V. When an auxiliary V is used, the reading of the drawing will be facilitated by turning the paper so as to bring G_1L_1 horizontal.

Auxiliary planes of projection perpendicular to V, and making an angle with H, may occasionally be used to advantage. In buildings and engineering structures, however, horizontal planes are the natural planes of stability; and the various parts of a structure are therefore laid out with respect to the horizontal ("level") and the vertical ("plumb"); hence the planes of projection are usually taken with respect to these two fixed directions. (Compare Art. 13.)

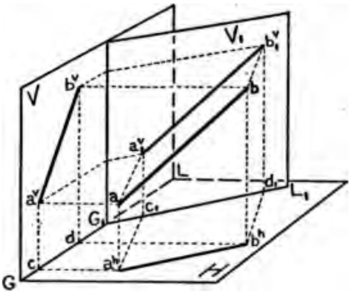

Fig. 190.

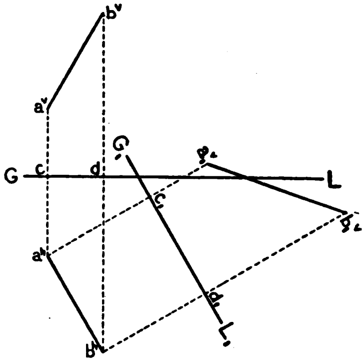

Fig. 191.

Problem 21. To find the projections and true length of the shortest distance from a given point to a given line.

(*a*) *First Method.* Let c, Fig. 192, be the given point, and A the given line. The shortest distance from a point to a line must be measured in a line containing the given point and perpendicular to the given line. Through the point c

pass a line, B, parallel to A. Find one trace, as HX, of the plane containing A and B. About HX as an axis, revolve into H the point c and the line A. (Revolve c; note that, if drawn, B would pass through s_1 and c_r; then draw A_r through s parallel to s_1c_r.) Having thus shown the actual relation of the given point and line, from c_r drop the perpendicular, c_rd_r, to A_r. Counter-revolve the point d, giving d^h and d^v. Draw c^hd^h and c^vd^v, the required projections of the shortest distance from point c to line A. The actual distance (true length) is the line c_rd_r.

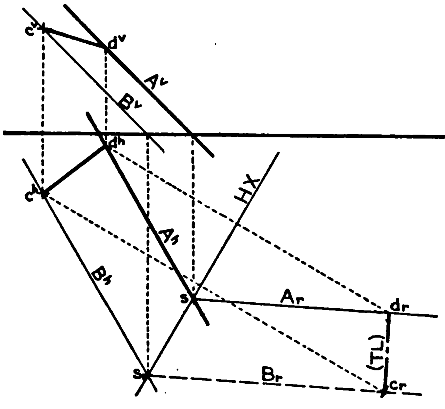

Fig. 192.

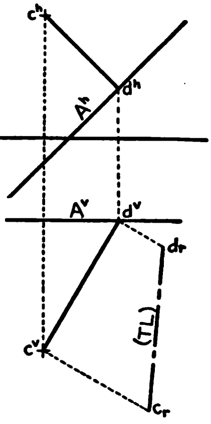

Fig. 193.

(*b*) *Special Case.* In the second example, Fig. 193, it will be seen that the given line A is parallel to H. The required shortest distance is a line perpendicular to A. Since A is parallel to H, the horizontal projection of any line perpendicular to A will be at right angles to A^h (Art. 30), hence the required horizontal projection, c^hd^h, may be drawn at once from c^h perpendicular to A^h. Project d^h to d^v and draw c^vd^v. In this case the actual distance must be found by obtaining the true length of cd, as shown at c_rd_r.

(*c*) *Second Method.* In the third example, Fig. 194, c (c^h, c^v) is the given point, and ab (a^hb^h and a^vb^v) the given line. The method consists in reducing the problem to Case (*b*) by means of a suitably chosen auxiliary *V*. Place the auxiliary ground line, G_1L_1, parallel to, and at any convenient distance from, a^hb^h. Find the auxiliary projection, $a_1^vb_1^v$, of the line *ab* (Art. 40); also the auxiliary projection, c_1^v, of point *c*. Since the given line *ab* is now parallel to the auxiliary *V*, the projection $c_1^vd_1^v$ may be drawn perpendicular to $a_1^vb_1^v$ (compare with Fig. 193). Project d_1^v to d^h, and draw c^hd^h; project d^h to d^v, and draw c^vd^v; c^hd^h and c^vd^v are the projections of the required shortest distance. To find the actual distance, find the true length, c_rd_r, of *cd*, as shown.

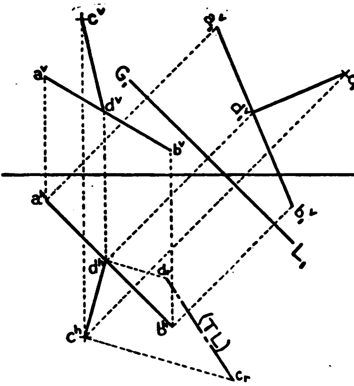

Fig. 194.

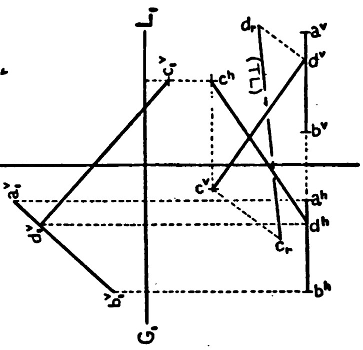

Fig. 195.

(*d*) In the fourth example, Fig. 195, the given line *ab* (a^hb^h and a^vb^v) is a profile line. The shortest distance from the given point *c* (c^h, c^v) to the line *ab* is found, as in the last example, by means of an auxiliary *V* taken parallel to the line *ab*. Note that, in this case, the auxiliary *V* is parallel to *P*, and gives the same view as a projection on *P* revolved into *H*.

Problem 22. To find the projections and true length of the shortest distance between two lines not in the same plane.

(*a*) *General Method.* Let *A* and *B*, Fig. 196, be the given lines. The shortest distance between any two lines must be measured in a line perpendicular to each of the given lines.

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Through either line, as A, using the auxiliary line D, pass a plane Q parallel to B (Prob. 15). Since B is parallel to Q, the projection of B on Q must be a line parallel to B. Find the projection, line L, of line B on the plane Q; that is, from any point e in B, draw the line F perpendicular to Q (Art. 33), and find the point, k, where F meets Q (Prob. 10, d); through k^h and k^v draw L^h and L^v respectively parallel to B^h and B^v.

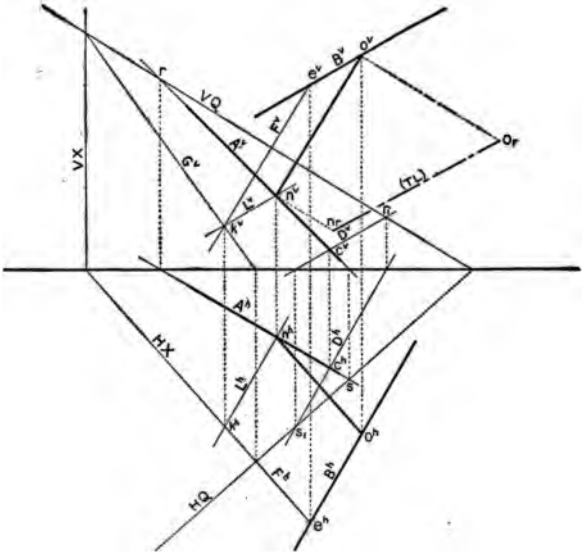

Fig. 196.

Lines A and L lie in the same plane, Q; hence they must intersect in a point, n. Since the lines B and L are parallel, the line ek may be moved in the plane of lines B and L until the point k falls at the point n in A. Draw no, parallel to ek, meeting B at o; then no, perpendicular to both B and A, is the required shortest distance. Find the true length of no, as shown at $n_r o_r$.

In working the problem, check the construction by noting if the ruled projectors from points n^h and o^h respectively pass through n^v and o^v.

Visualize each step in the construction.

(*b*) *Method by means of an auxiliary V.* Let it be required to find the projections and true length of the shortest distance between the lines *A* and *B*, Fig. 197. Find the horizontal trace, *HQ*, of a plane passed through line *A* parallel to line *B*. Find the vertical trace and *edge view* of the plane *Q* on an auxiliary *V* perpendicular to *Q;* that is, locate the projection c_1^v of the point *c* lying in plane *Q*, and draw the trace V_1Q through c^v and s_1. Assume any point, as *e*, on *B*, and draw its vertical and auxiliary vertical projections. From point *e* drop a perpendicular to plane *Q;* that is, draw e^hk^h and $e_1^vk_1^v$ respectively perpendicular to the traces *HQ* and V_1Q. Since V_1Q is an edge view of the plane *Q*, the line *ek* meets *Q* at the point k_1^v. Find k^h; locate k^v, and draw e^vk^v. As in the preceding case, the projection of line *B* on plane *Q* is the line *L*, drawn through point *k* parallel to line *B*. From the point *n*, where *L* intersects line *A*, draw the required line, *no*, parallel to *ek*. Since *no* is parallel and equal to *ek*, the true length of *no* is equal to $e_1^vk_1^v$.

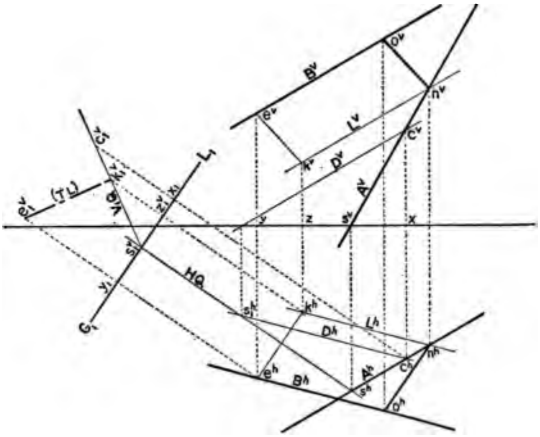

Fig. 197.

In working the construction, the same checks apply as in the preceding case.

(*c*) *Special Case.* When it is possible to obtain readily an auxiliary projection in which one of the given lines appears as a *point*, the construction can be simplified. Let A and B, Fig. 198, be the given lines, the line A being parallel to H. Draw the auxiliary ground line, G_1L_1, perpendicular to A^h, and find the projections of lines A and B on V_1; since line A is perpendicular to V_1, its projection on this plane will be the *point* A_1^v. As line A is perpendicular to V_1, any line perpendicular to A will be parallel to V_1; and, if a line parallel to V_1 is perpendicular to line B, it will project on V_1 at right angles to B_1^v (Art. 30). Hence line $n_1^v o_1^v$, drawn from A_1^v perpendicular to B_1^v, is the required shortest distance. The horizontal projection of line no is line $n^h o^h$, drawn through point o^h perpendicular to A^h (Art. 30). Project and draw $n^v o^v$, thus completing the required projections of the shortest distance between lines A and B.

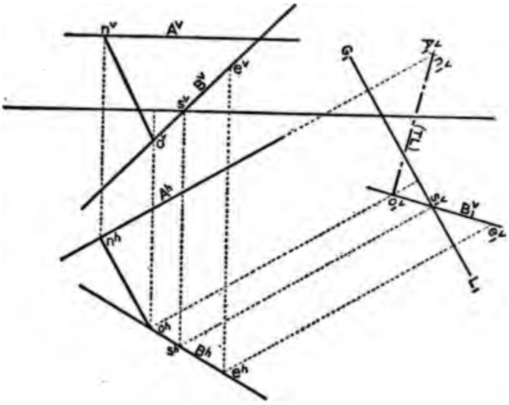

Fig. 198.

Problem 23. To find the angle between a given plane and either co-ordinate plane.

(*a*) Let it be required to find the angle between H and the plane Q, Figs. 199 and 200. The angle between any two planes is measured in a plane perpendicular to each (Art. 24); that is, in a plane perpendicular to the line of intersection of the given planes. Assume any auxiliary plane, *X*, perpendicular to the intersection, *HQ*, of *H* and the plane *Q*. The portion of the plane *X* included between *H*, *V*, and *Q* is the right triangle *abc*, having *VX* as altitude, *HX* as base, and *ac*, in *Q* (a line of maximum inclination to *H*, Art. 32), as hypothenuse. The angle between *H* and *Q* is the angle α between the hypothenuse and base of the triangle *abc*. Hence, to find the true size of the angle α, revolve the triangle *abc* into either *V* or *H*, as shown.

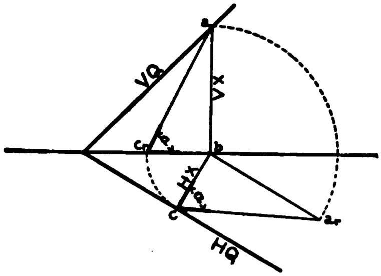

Fig. 199.

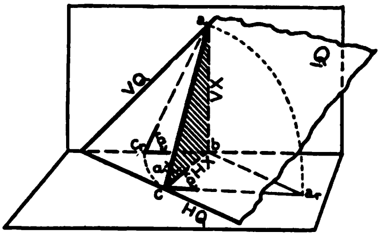

Fig. 200.

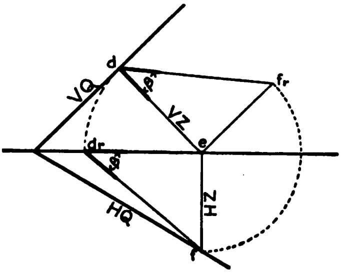

Fig. 201.

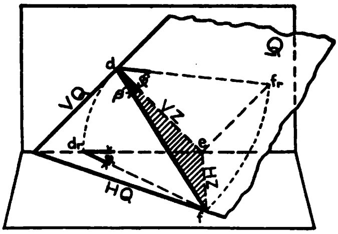

Fig. 202.

To find the angle between *V* and the plane *Q*, Figs. 201 and 202, assume any auxiliary plane, *Z*, perpendicular to the intersection, *VQ*, of *V* and the plane *Q*. Revolve into either *H* or *V* the triangle included between *V*, *H*, and *Q*, giving the required angle β, as shown.

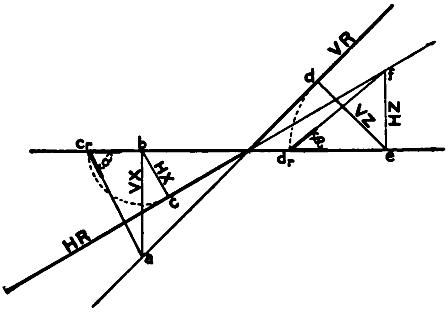

Fig. 203.

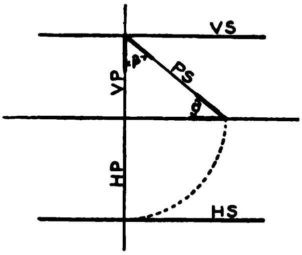

Fig. 204.

(*b*) In the second example, Fig. 203, are found the angle, α, between *H* and the plane *R;* also the angle, β, between *V* and *R*.

(*c*) *Special Case.* Let the plane *S*, Fig. 204, be parallel to *GL*. In this case a plane perpendicular to *H*, *V*, and *S* is evidently a profile plane. Assume any auxiliary profile plane, *P*, and revolve the triangle lying in *P* and included between *H*, *V*, and *S* into either *V* or *H*, giving the required angles α and β as shown.

Problem 24. Given one trace of a plane, and the angle which the plane makes with either co-ordinate plane, to find the other trace of the given plane. (Two cases.)

Case I. Given the trace on one co-ordinate plane and the angle which the plane makes with the same *co-ordinate plane.* Let *HQ*, Fig. 205, be the given trace, and α the given angle which the plane *Q* makes with *H*. The construction of this problem is the reverse of the preceding one. Assume any auxiliary plane, *X*, perpendicular to the given intersection, *HQ*. Revolve the trace *HX* about *VX* into *V*, as shown at $c_r b$. Upon the line c_rb as a base, using the given angle α, construct ac_rb, the revolved position of the triangle lying in X and included between H, V, and the required plane Q. Imagining the triangle counter-revolved about VX, it will be seen that the edge ac pierces V in point a; hence through points a and n draw VQ, the required trace of the plane Q. (Compare with Fig. 199.) It will be seen that this plane slopes downward and *forward.*

Repeating the same construction below GL, the line $a'c$ pierces V at a', giving a second solution. Through points a' and n draw VQ', the required trace of a plane sloping downward and *backward.* (Compare with Fig. 203.)

Case II. Given the trace on one co-ordinate plane and the angle which the plane makes with the other *co-ordinate plane.* Let VQ, Fig. 206, be the given trace, and α be the given angle with H. Draw the vertical trace, VX, of an auxiliary plane

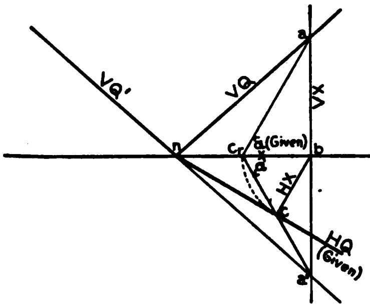

Fig. 205.

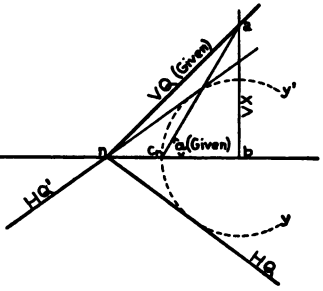

Fig. 206.

perpendicular to H. With ab as a side, construct on GL the right triangle abc_r, containing at c_r the given angle α. Imagining the triangle abc revolved about VX as an axis, the point c will move in the arc yc_ry', described with b as a centre. Comparing with Fig. 199, it will be seen that HQ must be tangent to this arc; hence from point n draw the required trace, HQ, tangent to the arc c_ry. It is evident that there is a second solution, HQ', drawn from point n tangent to the arc c_ry'.

Problem 25. Given the angles which a plane makes with H and V, to find the traces of the plane.

Let it be required to find a plane, Q, making a given angle, α, with H, and a given angle, β, with V. (The sum of α and β must not be less than 90° nor more than 180°.)

(*a*) Consider the problem solved, and let Q, Fig. 207, be the required plane. Assume any point o in GL, and through o pass the line A, lying in V and perpendicular to H. About the line A as an axis revolve the plane Q until it is perpendicular to V (Art. 35, *d*). The new vertical trace, V_1Q, is an *edge view* of the plane, and makes with GL the given angle, α, which the plane Q makes with H. From o drop the perpendicular, ot_r^v, to V_1Q. Since V_1Q is an edge view of the plane Q, ot_r^v must lie in V, and is, therefore, the true perpendicular distance from the point o to the plane Q.

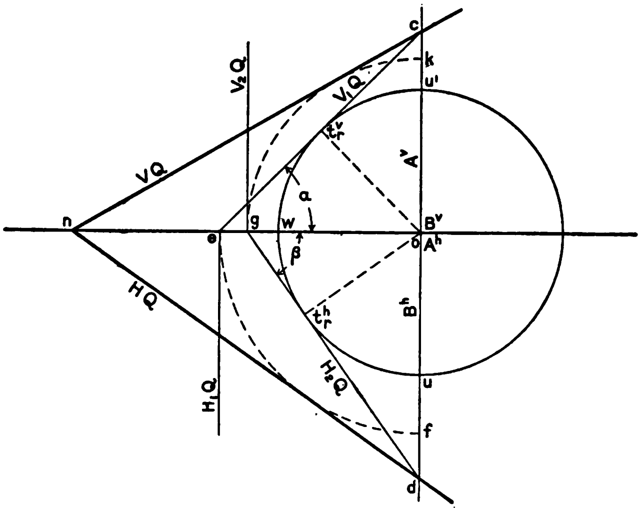

Fig. 207.

Through o pass also the line B, lying in H and perpendicular to V. Revolve the plane Q about line B as an axis until it is perpendicular to H. The new horizontal trace, H_2Q, is an edge view of the plane Q, and makes with GL the given angle, β, which the plane Q makes with V. From o drop the perpendicular, ot_r^h, to H_2Q. This line lies in H, and is the true perpendicular distance from the point o to the plane Q. But, as there can be only one perpendicular from a point to a plane, ot_r^h and ot_r^v are the true lengths of the same line, ot, in space, passing through o and perpendicular to the plane Q; whence it follows that H_2Q and V_1Q are both tangent to the circle uwu' (composed of the semi-circles uw in H and wu' in V) drawn with o as centre, and $ot_r^h = ot_r^v$ as radius.

(*b*) *Construction.* With any point, o, in GL, as centre, and any assumed radius, describe the circle uwu'. Through o draw the axes A and B perpendicular to GL.

Draw V_1Q—representing the plane Q perpendicular to V—tangent to the circle uwu', and making the given angle, α, with H. Similarly, draw H_2Q—representing the plane Q perpendicular to H—tangent to the circle uwu', and making the given angle, β, with V.

If the plane Q be revolved about the axis A, the V-trace of each successive position of Q inclined at α with H must pass through the intersection, c, of A^v and V_1Q. Similarly, if Q be revolved about the axis B, the H-trace of Q must pass through the intersection, d, of B^h and H_2Q.

When Q is perpendicular to V, its horizontal trace, H_1Q, is perpendicular to GL, and tangent to a circle, described from o, whose radius varies according to the angle which V_1Q makes with GL; for the given angle, α, with H, the circle is ef, and its radius oe. Since H_1Q is perpendicular to oe, it is evident that, when Q is revolved about A, the H-trace of Q must take successive positions always tangent to the arc ef, and for *some position* of Q will pass through the fixed point, d. But it has been shown that, when Q is inclined at the angle β with V, HQ must pass through d; hence a line drawn through d, and tangent to the arc ef, must be the horizontal trace of Q inclined both at α with H and at β with V. By similar reasoning it can be shown that a line drawn through the point *c*, and tangent to the circle *gk*, must be the vertical trace of *Q*. Draw the arcs *ef* and *gk*, and the resultant traces.

Note. The lines *HQ* and *VQ* must intersect in *GL*.

(*c*) *Special Case.* Let the sum of the given angles, α and β, be 90°. In this case the required plane, *Q*, must be parallel to *GL*. Assume any auxiliary profile plane, *P*, Fig. 208. Draw the profile trace, *PQ*, of the required plane, making the given angles with *H* and *V*. Counter-revolve *PQ*, and draw *HQ* and *VQ*, the required traces of the plane *Q*.

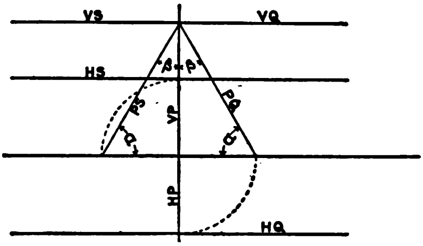

Fig. 208.

(*d*) *Additional Results.* In Fig. 207 the result is the plane *Q*, sloping downward, forward, and to the left. Besides the plane *Q*, three other non-parallel planes are possible, each making the given angles with *H* and *V*. Representing the four planes by the planes *Q*, *R*, *S*, and *T*, Fig. 209, the general directions of the planes are shown by the respective directions of *A*, *B*, *C*, and *D*, lines of maximum inclination with *H*. If it were required to find the four non-parallel planes making the same angles with

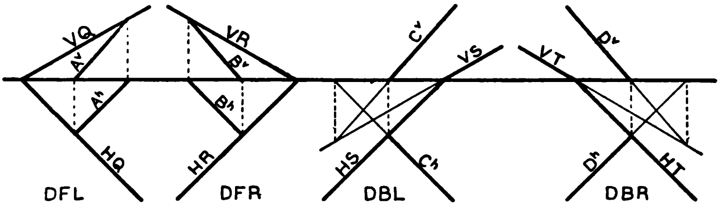

Fig. 209.

H and *V*, a construction similar to that of Fig. 207 might be repeated in each quadrant. A simpler method, however, is shown in Fig. 210. Constructing plane *Q* by the method of Fig. 207, it will be seen that *Q* passes through the points *c*, *n*, and *d*. Make *on′* equal to *on*, and pass plane *R* through *c*, *n′*, and *d*. Make *od′* equal to *od*, and pass plane *S* through *c*, *n*, and *d′*; also pass plane *T* through *c*, *n′*, and *d′*. Four other planes may be similarly obtained, but they will be parallel to the four already found.

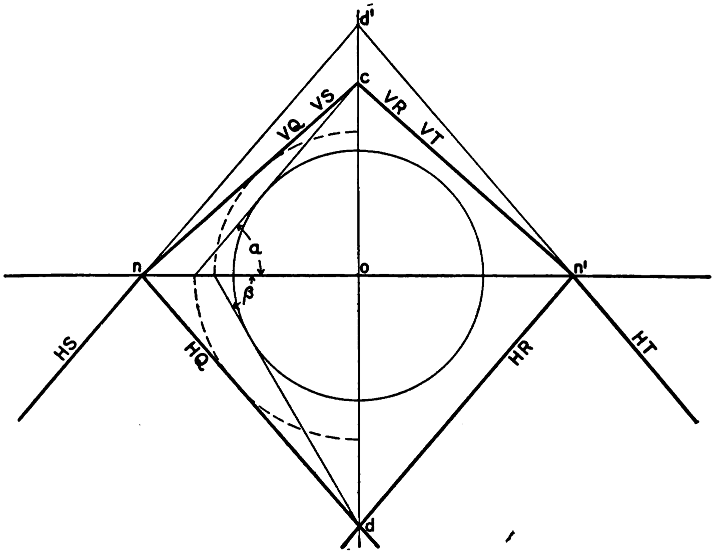

Fig. 210.

In Fig. 208, besides the plane *Q*, sloping downward and forward, a second plane *S*, sloping downward and backward, may be obtained, as shown. In this case only two non-parallel planes are possible.

Problem 26. To find the true size of the angle between two given planes.

(*a*) Let it be required to find the true size of the angle between the planes *Q* and *R*, Fig. 211. The angle between two planes is measured in a plane, Z, perpendicular to the line of intersection of the given planes. Find the intersection, A, of the planes Q and R (Problem 9). In any convenient position take an auxiliary V parallel to the intersection A; that is, draw G_1L_1 parallel to A^h. Project A upon the auxiliary V, as shown by A_1^v. Pass the plane Z perpendicular to A (Art. 33), intersecting A at point c_1^v. Project c_1^v to A^h, and draw c^hd^h and c^he^h, giving $d^hc^he^h$, the horizonal projection of the angle between Q and R. To find the true size of the angle, revolve the angle dce into H, as shown.

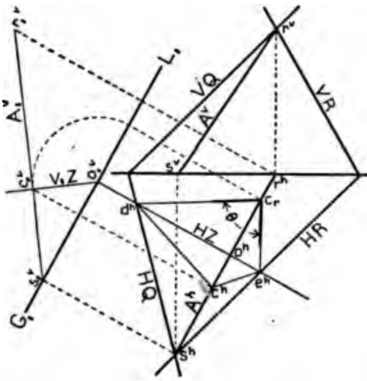

Fig. 211.

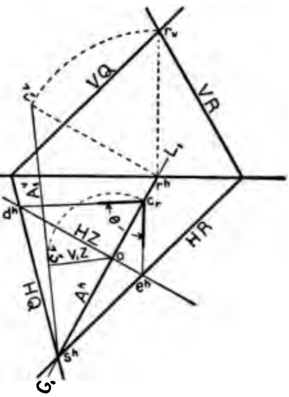

Fig. 212.

(*b*) *Working Method.* Let the given planes, Q and R, Fig. 212, be the same as in Fig. 211. The working method or practical construction consists in taking G_1L_1 coincident with A^h; that is, in passing the auxiliary V through A. In determining the auxiliary vertical projection of A, since the height of a point, r, in A is already determined, it is unnecessary to find the vertical projection of A. Also, since the true angle depends directly upon the revolution of c_1^v, Fig. 211, the projection $d^hc^he^h$ of the angle is unnecessary. Otherwise it will be seen that the constructions of Figs. 212 and 211 are similar.

(*c*) *Additional Examples.* Let it be required to find the angle between the planes S and T, Fig. 213, intersecting GL at the same point. Find the intersection, A^h, by means of the auxiliary plane X (Problem 9, Case F). The height of the

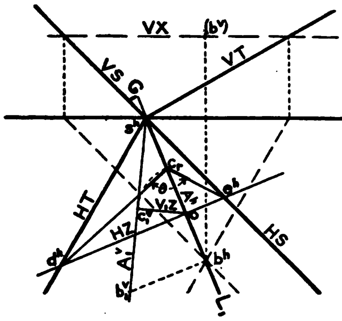

Fig. 213.

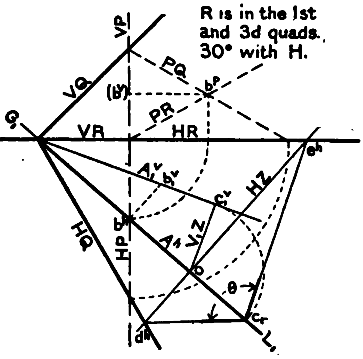

Fig. 214.

point b^v above H is evidently the same as the distance from GL to VX, and the construction is similar to that of Fig. 212.

In Fig. 214 the given plane R passes through GL. One point, b, of the line of intersection, A, is obtained by means of the auxiliary profile plane P (Problem 9, Case G). The position of A_1^v is determined by means of the point b_1^v, its distance from H being the same as the distance from GL to b^p. Observe that the rest of the construction is similar to that of Fig. 212.

(*d*) *First Special Case. When the H- or V-traces are parallel.* Let Q and R, Fig. 215, be the given planes, and the traces HQ and HR parallel. The intersection, A, is an H-parallel common to Q and R. Take an auxiliary V perpendicular to A^h. Since A is parallel to H, it is perpendicular to the auxiliary V; which must therefore be perpendicular to Q and R, and contain the angle between them. Revolving the auxiliary V into H, make oc_r equal to the distance from GL to A^v. Draw dc_r and ec_r, giving the required angle θ.

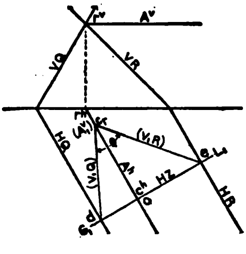

Fig. 215.

(*e*) *Second Special Case. When one of the given planes is parallel to a co-ordinate plane.* The required angle is equal to the angle between the same co-ordinate plane and the second given plane; it is most readily found according to Problem 23.

Problem 27. Given a plane, and a point in the plane revolved into H or V, to find the projections of the point.

(*a*) Let *Q*, Fig. 216, be the given plane, and a_r^h the given point; it is required to counter-revolve a_r^h into *Q*. In its counter-revolution, the point a_r^h must move about *HQ* as an axis, and

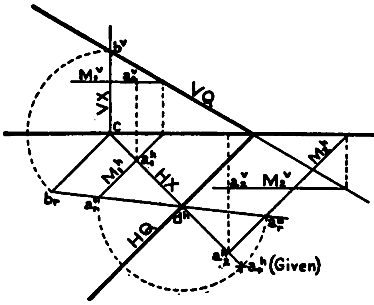

Fig. 216.

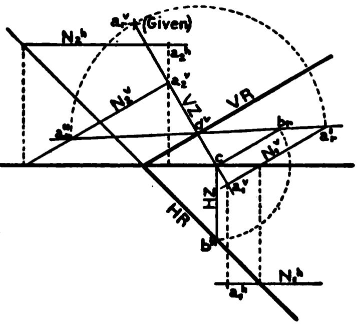

Fig. 217.

in the plane *X* perpendicular to *HQ*. The path of a_r^h is a circular arc lying in *X;* the centre of the arc lies in *HQ*, and the point where the moving a_r^h meets the plane *Q* is the intersection of the arc with the line common to *X* and *Q*. In order to represent these conditions, revolve *X* about *HX* into *H*, as shown by the revolved position of the triangle d^hcb_r, contained in *X* and included between *H*, *V*, and *Q*. It will be seen that the line b_rd^h is the revolved position of the line common to *X* and *Q*, and represents the portion in the first quadrant; also that b_rd^h produced, $d^ha''_r$, represents the line produced into the fourth quadrant. Further, if *HQ* be imagined revolved with *X*, when *X* meets *H*, *HQ* is seen as the point d^h. Hence, with d^h as centre, describe the arc a''_r a'_r through a_r^h. The portion $a_r^ha'_r$ of the arc represents the given point counter-revolved *upward*, and therefore as meeting the plane *Q* at a'_r in the first quadrant; while the portion $a_r^h a''_r$ of the arc represents the given point counter-revolved *downward*, and therefore as meeting *Q* in the fourth quadrant. Counter-revolve a'_r and a''_r about *HX*, giving the required horizontal projections a_1^h and a_2^h respectively. To find the vertical projections of the points, use the H-parallels M_1 and M_2, as shown (Problem 7).

(*b*) In the second example, Fig. 217, let *R* be the given plane, and a_r^v the given point. In its counter-revolution the given point must move in the plane *Z*, perpendicular to *VR*. The revolved position, $a''_r a'_r$, of the line common to *Z* and *R*, is the hypothenuse, produced, of the revolved triangle $d^v c b_r$, contained in *Z* and included between *H*, *V*, and *R*. It will be seen that $d^v b_r$ represents the portion of the line in the first quadrant; $b_r a'_r$, the portion in the fourth quadrant; and $d^v a''_r$, the portion in the second quadrant. Hence, counter-revolving a_r^v *forward* gives a'_r, which, counter-revolved about *VZ*, gives the required projection, a_1^v, in the fourth quadrant. Counter-revolving a_r^v *backward* gives a''_r, which, counter-revolved about *VZ*, gives the required projection, a_2^v, in the second quadrant. To find the horizontal projections of *a*, use the V-parallels N_1 and N_2, as shown (Problem 7).

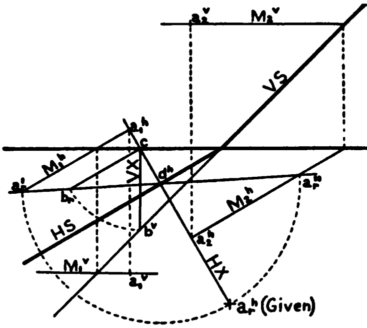

Fig. 218.

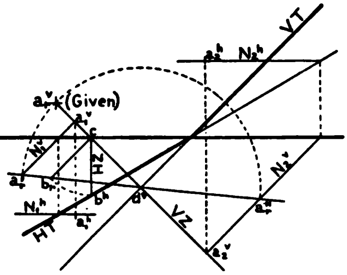

Fig. 219.

(*c*) Additional examples are given in Figs. 218 and 219. Compare with Figs. 216 and 217 respectively.

(*d*) In Fig. 220 it is required to counter-revolve the given plane figure $e_r f_r g_r k_r$ about HQ into the portion of plane Q, which lies in the first quadrant. Through any corner, as e_r, of the given figure, pass the plane X perpendicular to HQ, and counter-revolve the point e_r according to paragraph *a*. Counter-revolve the remaining corners of the figure by means of H-parallels, as illustrated by line M through point g.

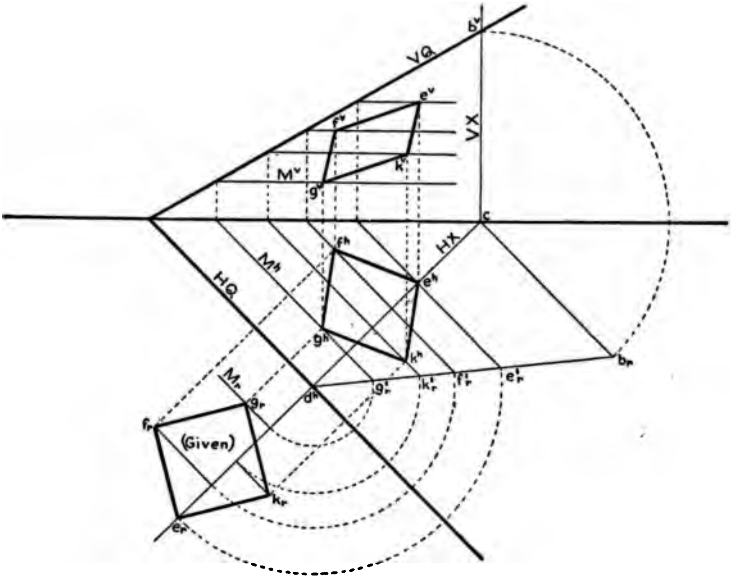

Fig. 220.

CHAPTER VII.

PROJECTION OF PLANE FIGURES AND SOLIDS; SECTIONS AND DEVELOPMENTS; INTERSECTION OF SOLIDS.

41. In practical drafting, ground lines are omitted, except when required by some special construction, and the terms H-, V-, and P-projections of descriptive geometry are replaced by the commonly used terms *plan*, *elevation*, and *side elevation*. This may be confusing to the beginner, and tend to obscure the connection between the theory of descriptive geometry and its applications. To accustom the student to their use, the practical terms will be introduced in this chapter, and the ground line frequently omitted.

42. The projections of a figure, plane or solid, may be regarded as of two kinds. First, *resultant projections*, obtained by projecting a figure of *known* shape, size, and position, as *E* and *F*, Fig. 224. Second, *assumed projections*, representing a figure of *unknown* shape, size, and position. In the latter case the projections may be drawn arbitrarily, or according to a description, not of the figure in space, but of its projections. Thus, for example, in Fig. 221, a circle is taken as the base of an oblique cone, and the plan and elevation of the cone are drawn as shown. It is evident that the cone lies in the third quadrant, with its axis inclined upward, backward, and to the right; but the true shape of the cone, its size, and the angles which the axis makes with *H* and *V* are unknown.

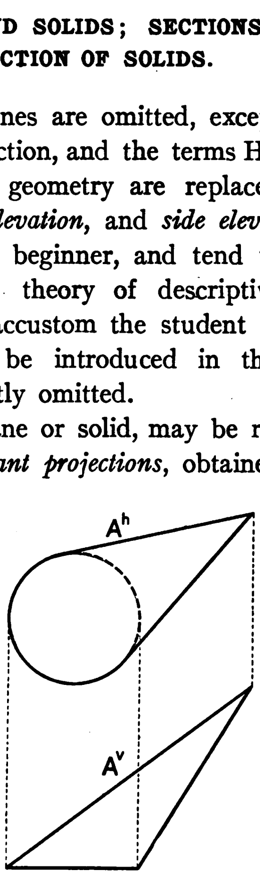

Fig. 221.

When, as in Fig. 222, the plan and elevation of a solid are so taken that the base is seen edgewise in one of the views, any number of corners or points in the base can be assumed at random; but if the base is inclined to both *H* and *V*, as in

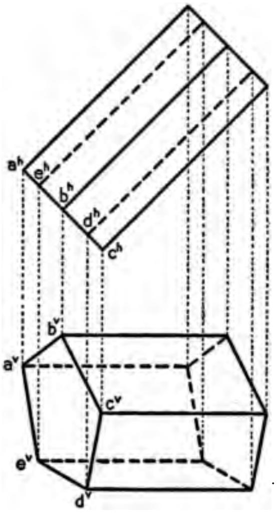

Fig. 222.

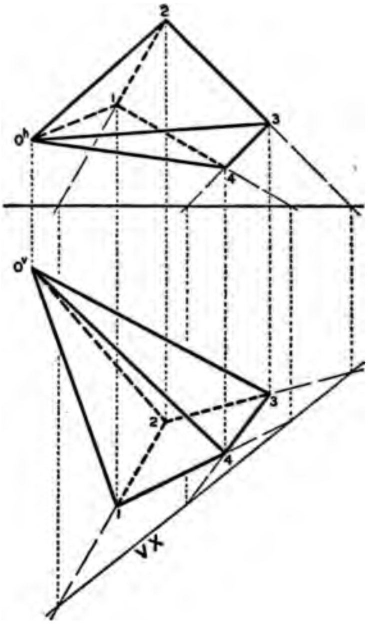

Fig. 223.

Fig. 223, only three corners, as 1, 2, and 3, can be assumed at random, since additional ones must be taken *in the plane* of the first three (Prob. 20).

43. In finding the projections of plane figures and solids, the draftsman should proceed systematically. For example, any extensive series of points should be projected in their order, and in many cases this order will be more quickly recognized in the different projections if the points are distinguished by numbers instead of letters. In the following constructions, numbers, letters, or both, will be used, according to convenience.

Construction 1. Given the true shape and size of a plane figure, to construct its plan and elevation when inclined at given angles with H and V.

Example 1. (Fig. 224.) Let the figure be a quadrilateral, ***kmno***, placed in the first quadrant and inclined at 45° with *H*.

(*a*) The true shape of the given figure must first be shown. Draw the plan, *A*, and elevation, *B*, of the figure lying in *H*.

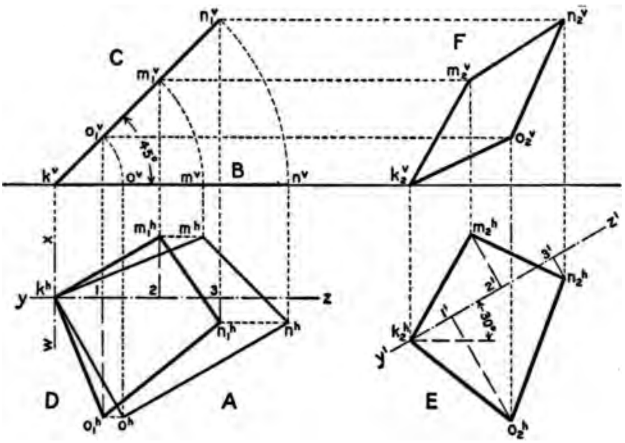

Fig. 224.

Revolve the figure about the axis *wx*, drawn through point *k* perpendicular to *V*, until the elevation, *C*, shows the required angle with *H*. In revolving about an axis perpendicular to *V*, the distance from *V* of each point in the figure does not change; hence the distance from *V* of each point in the plan *D* can be obtained by projecting horizontally from the plan *A*. Project also from the elevation *C* and draw the required plan *D*.

(*b*) Let it now be required to find the elevation of the quadrilateral after the plan *D* has been revolved through an angle of 30°. For clearness, the plan is transferred to *E*, the transfer being made by means of the base line, *yz*, and ordinates as shown. In this revolution and transfer, the distances of the points from *H* do not change; hence the distances from *H* of the points in the elevation *F* can be obtained by projecting horizontally from the elevation *C*. Project also from the plan *E* and complete the required elevation *F*.

Example 2. In Fig. 225 are found the projections of a circle in the third quadrant. The circle is first placed parallel to *V*

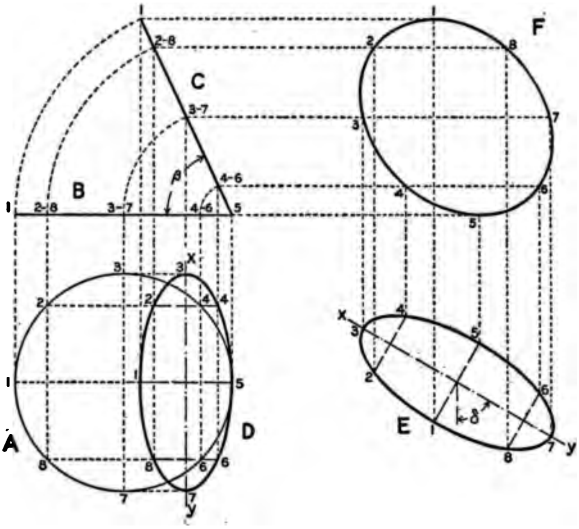

Fig. 225.

(elevation *A* and plan *B*). It is then revolved to make an angle β with *V* (plan *C* and elevation *D*). Finally, the elevation *D* is revolved through the angle δ, transferred to *E*, and the plan, *F*, is constructed.

Note that the projections are obtained by means of assumed points on the original circle. The number of points assumed in Fig. 225 is sufficient to illustrate the method; in a large figure however, or where great accuracy is required, additional points should be taken, especially at the sharper portions of the curve. Note also that the assumed points are distinguished by numbers instead of by letters (Art. 43) and that, in accordance with drafting practice, the ground line, and the index letters *h* and *v*, are omitted.

Construction 2. Given the true shape and size of a prism or pyramid, to construct the plan and elevation when its axis is inclined at given angles with H and V.

First Method. Let the solid be a square right prism, having its lateral edges inclined at an angle α with H and β with V. The prism is placed in the first quadrant, and one corner of the base rests on H at a given point, a.

Since the lateral edges of the prism make angles of α with H and β with V, the base is inclined at the complementary angles, $90^\circ - \alpha$ with H and $90^\circ - \beta$ with V. In any position,

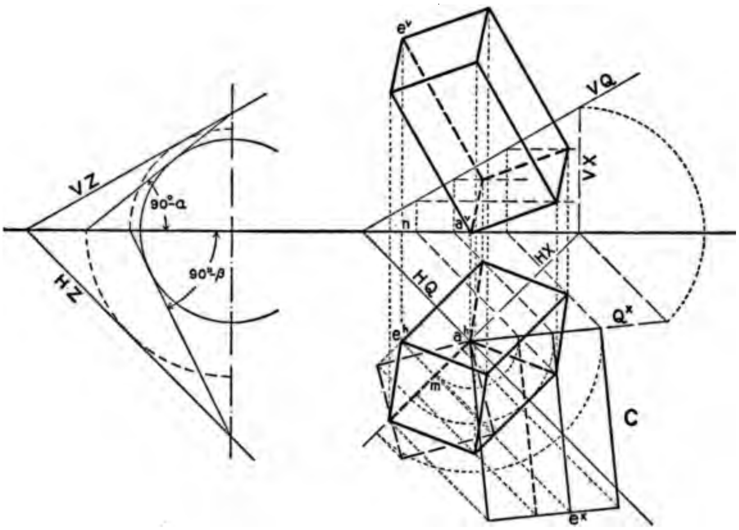

Fig. 226.

Fig. 226, find the traces of a plane Z, parallel to the base of the prism (Prob. 25). Parallel to Z, and passing through the given point, a, draw Q, the plane which contains the base. Next, find the plan and elevation of the base of the prism lying in the plane Q (Prob. 27). Draw the lateral edges of the prism perpendicular to the base, seen edgewise in Q^x, giving the said view C. Project from the view C, and draw the plan. Draw

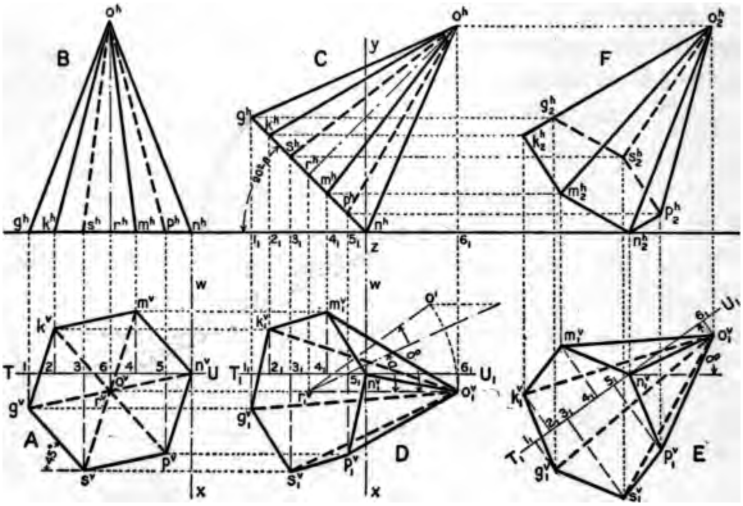

Fig. 227.

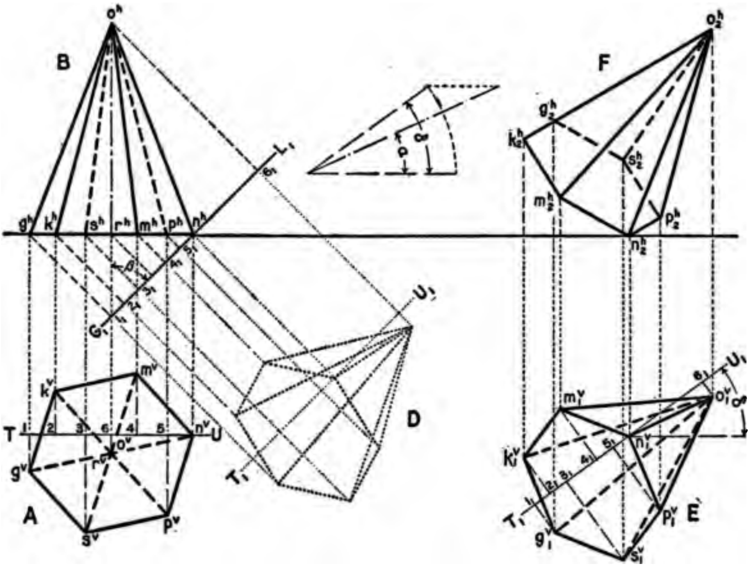

Fig. 228.

the elevation by projecting from the plan, and making the heights from H the same as represented in the projection C ($e^v n = e^x m$).

SECOND METHOD. Let the solid be a hexagonal, right pyramid, having its axis inclined at an angle α with H and β with V. The pyramid is placed in the third quadrant, and one corner of the base touches V at a given point, n.

(*a*) Consider the pyramid first placed with its base lying in V, and two basal edges making 45° with H, as shown by the elevation A and the plan B, Fig. 227.

(*b*) About an axis wx, perpendicular to H and passing through the fixed corner, n, of the base, revolve the pyramid until its axis makes the given angle, β, with V. (Place the base at the angle $90° - \beta$ with V, and for greater clearness let the revolution be shown at C and D instead of upon A and B.)

In revolving about the axis wx, the distance from H of each point in the pyramid does not change. Hence the distance from H of each point in the elevation D can be obtained by projecting horizontally from the elevation A.

(*c*) About a second axis, yz, perpendicular to V and passing through the fixed corner, n, of the base, revolve the pyramid until its axis makes the given angle, α, with H. That is, by drawing on the elevation D, find the angle, δ, through which the elevation of the axis of the pyramid must be turned, so that it shall represent the axis of the pyramid in space when inclined at the given angle, α, with H. Having determined (Prob. 3) the new direction, $r_1^v o'$, of the elevation of the axis, the point n may, for greater clearness, be transferred to n_1^v, E. Through this point, draw the base line, $T_1 U_1$, parallel to $r_1^v o'$. Transfer from D the co-ordinates of the vertex and the corners of the base, and complete the elevation E.

In revolving the pyramid about the axis, yz, the distance from V of each point in the pyramid does not change. Hence the distances from V of the points in the plan F can be obtained by projecting horizontally from the plan C.

Visualize the several positions of the pyramid in space, and note which edges of the solid are visible.

THIRD OR WORKING METHOD. This is an abridgment of the second method. The same data are used, and the method will be best understood through a comparison of Figs. 227 and 228.

Instead of revolving the plan B, Fig. 227, to make the given angle, β, with V, take an auxiliary V, Fig. 228, making with the axis of the plan B the given angle β. Project the plan B upon G_1L_1. It will be noted that the distances 1_1–2_1 2_1–3_1 3_1–4_1, etc., on G_1L_1 between the ruled projectors, are the same as the correspondingly numbered distances on GL, C, Fig. 227. But these are the same as the distances between the offsets drawn on the elevation D, and used in constructing the elevation E. Also, Fig. 227, the lengths of the offsets from T_1U_1 in E, transferred from D, are the same as from TU in A. Hence, Fig. 228, having determined (Prob. 3) the direction of the elevation of the axis, or a base line parallel to it, the elevation E may be constructed directly by means of a base line and offsets determined by measurements taken from G_1L_1 and the elevation A. The distances from V (projected from C in Fig. 227) required in drawing the plan F are obtained in Fig. 228 by measuring the ruled projectors between G_1L_1 and the plan B. It will be seen, then, that the present method eliminates the projections C and D, Fig. 227. The elevation shown at D, Fig. 228, corresponds to D, Fig. 227, and is introduced merely as a first aid in visualizing the construction.

Visualize the construction, including all lines upon which measurements are taken.

Construction 3. To draw the plan and elevation of a prism parallel to one co-ordinate plane, oblique to the other, and having one face inclined at a given angle with the plane to which the prism is parallel.

Let the prism be triangular, and its bases perpendicular to the lateral edges. It is placed in the third quadrant, parallel to H, inclined at 45° with V, and its top lateral face makes 15° with H.

Draw, Fig. 229, the plan $d^h m^h$, and elevation $d^v m^v$, of the upper edge of the top face. Since the edges of the prism are

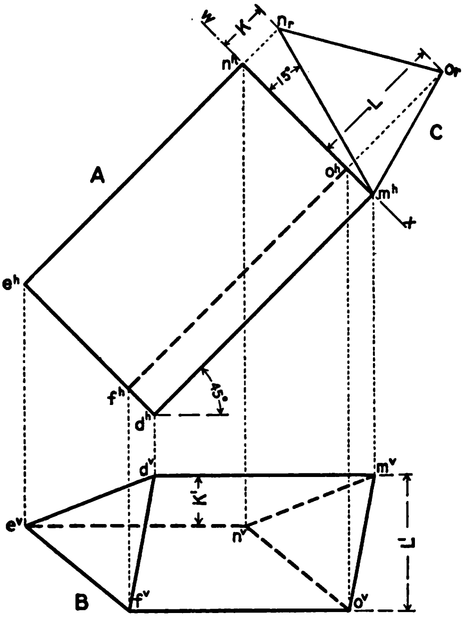

Fig. 229.

parallel to H, the plan and elevation of the prism can be constructed by means of an end view, C, obtained by supposing that the prism is revolved until perpendicular to H, about the horizontal axis, wx, passing through m and perpendicular to dm. The line $m^h n_r$ is an edge view of the top face of the prism, hence it must be placed at 15° with wx. The vertical distances K' and L' between the elevations of the lateral edges are the same as K and L in the end view C.

Construction 4. Given the projections of a solid, to revolve the solid about an axis parallel to H or V.

EXAMPLE 1. Given the assumed plan, A, and elevation, B, of an irregular triangular pyramid, *ogkm*, placed in the third quadrant as shown, Fig. 230. The pyramid is truncated by the plane, *Q*, shown in the side elevation, *C*, parallel to *GL* and making 30° with the base of the solid. It is required to find the plan, *E*, and elevation, *F*, of the pyramid and section, when the former is revolved, backward, through 60°, about an axis taken through the corner, *k*, of the base and parallel to *GL*.

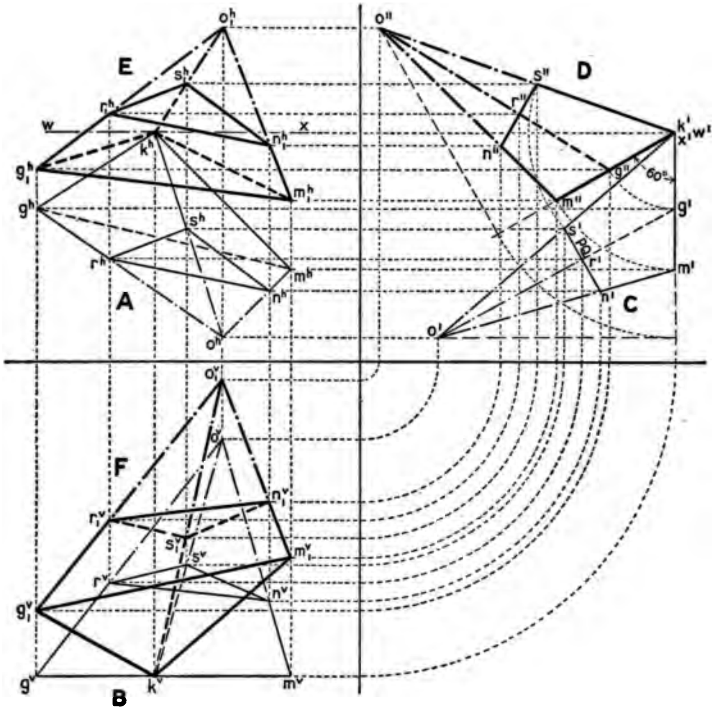

Fig. 230.

Since the axis of revolution *wx* is parallel to *GL*, the revolution of the pyramid must first be shown on a plane of projection perpendicular to the axis; that is, by a side elevation, or projection on *P*, constructed from the given plan and elevation.

Observe the construction. Visualize the pyramid in space. Note which surfaces and edges are visible.

EXAMPLE 2. Given, Fig. 231, the altitude, *no*, and the plan, *A*, of a pentagonal right pyramid placed in the first quadrant with its base resting on *H*. It is required: *First*, to draw the

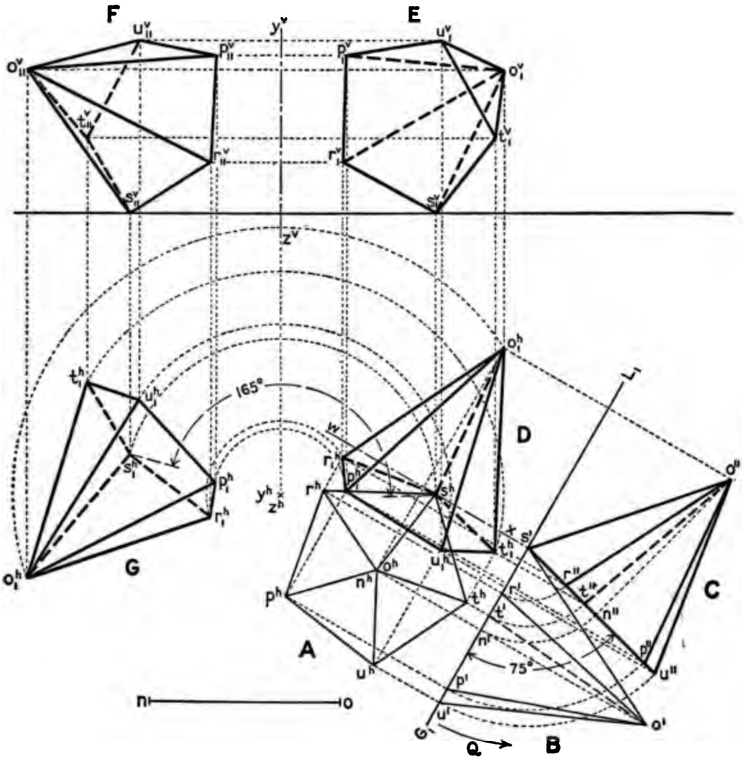

Fig. 231.

plan, *D*, and elevation, *E*, of the solid when revolved through 75°, backward to the right, in the direction *Q*, and about an axis, *wx*, taken through the corner, *s*, of the base. *Second*, to find the elevation, *F*, when the pyramid is revolved through 165° about an axis, *yz*, perpendicular to *H*, and taken outside of the pyramid.

Observe the construction. Visualize each position of the solid in space. Note which surfaces and edges are visible.

Construction 5. Given the true shape and size of a curved solid, to draw its projections when inclined to H and V.

Example 1. Let the solid be a right cone, placed in the third quadrant, having its axis inclined at 30° with *H*, and 37½° with *V*. The construction, Fig. 232, is essentially the same

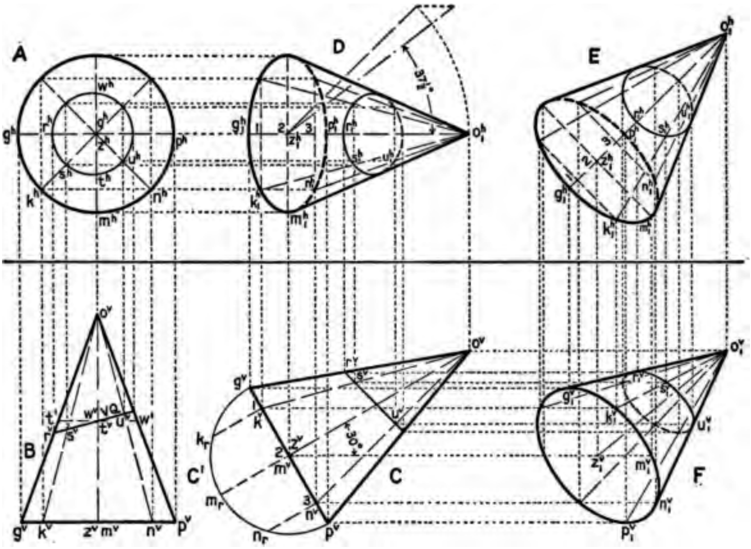

Fig. 232.

as that of the pyramid, Fig. 227, except that the base must be projected by means of assumed points, as in Fig. 225.

If the projections of a section of a cone (or cylinder) are required, they may be found by means of assumed elements. Thus, for example, in Fig. 232, the several projections of the section are obtained by connecting the projections of the points, *r*, *s* . . . *u* (first determined on *A* and *B*), where the plane *Q* intersects the assumed elements, *og*, *ok* . . . *on*. The width, $t^h w^h$, in *A*, not obtainable by projecting from *B*, is equal to $t'w'$.

Working Method. The projections *A* and *B*, Fig. 232, are unnecessary. At *C*, the elevation is drawn with the axis inclined at 30° with H, and the plan A is replaced by the half base, C', revolved parallel to V. The axis about which the half base is assumed to be revolved is gp, the diameter of the base parallel to V, and projected in the plan D at $g_1^h p_1^h$ parallel to GL. The ordinates k_r1, m_r2, and n_r3, drawn through the assumed points on the revolved half base and intersecting $g^v p^v$, represent lines actually perpendicular to V, consequently parallel to H. Hence, projecting to $g_1^h p_1^h$ and drawing the plan of the ordinates perpendicular to $g_1^h p_1^h$, the true lengths of the ordinate can be transferred from C'.

Visualize the several positions of the solid in space, including the assumed ordinates and elements. Note in each position of the solid which elements and what portion of the outline of the base are visible.

Example 2. Let it be required to draw the projections of a torus, placed in the third quadrant, and inclined to H and V.

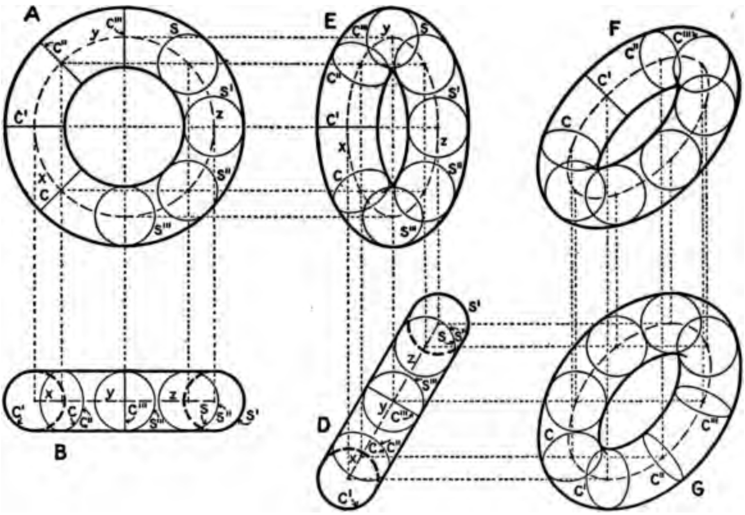

Fig. 233.

If the torus be regarded as generated by the revolution of a circle about a straight line outside the circle, the required projections of the torus will be composed of lines tangent to ellipses, as C, C'', and C''', F, Fig. 223, representing different positions of the generating circle. This construction, however, is laborious, and an accurate result difficult to obtain.

A much simpler construction will result if the torus be regarded as generated by the revolution of a sphere instead of a circle. Every portion of the visible contour of a torus, in whatever direction the solid is viewed, is tangent to some projection of the generating sphere. But the sphere always projects as a circle; hence, if the projections of assumed positions of the centre of the generating sphere be found, it only remains to describe circles, as S, S', S'', and S''', from these points, and, tangent to them, to draw the contours of the projections of the torus. The centre line xyz is not necessary to the construction, but suggests more clearly the path of the centre of the sphere.

44. Developments. A development is a drawing giving the true shape, size, and relative position of the surfaces of a solid, spread out in the same plane. When accurately cut out of thin sheet metal or cardboard, and properly curved or bent, a development practically reproduces the solid.

Construction 6. To draw the development of a solid bounded by plane faces.

Given the assumed projections of a pyramid, Fig. 234, it is required to develop the frustum obtained by passing through the pyramid the plane X.

A triangle is determined and readily plotted when its sides are known; hence, in developing a solid bounded by plane surfaces, a common method is to reduce the surfaces to a system of triangles, the development being composed of the true shape and size of the triangles.

In the present case of the frustum the lateral faces are quadrilaterals. The lateral faces of the original pyramid, however, are triangles, and the vertex is accessible. It is therefore best first to develop the lateral surfaces of the pyramid, and then to find the portion of these surfaces bounding the frustum. The true size of the bases of the frustum may then be constructed by means of triangles, obtained by drawing a diagonal of each base.

Construction, Fig. 234. Find, as needed, the true lengths of the lateral edges of the pyramid, the portions of these edges included between the bases of the frustum, and the diagonal *mp* of the upper base. Using the true lengths of the edges of the face *ode*, construct the triangle $o'd'e'$ of the development. Draw the triangles representing the true sizes of the remaining faces of the pyramid, as shown. Lay off the true lengths of the lateral edges of the frustum, and complete the development of its lateral faces. Add the bases $d''e''f'g'$ and $k''m''n'p'$.

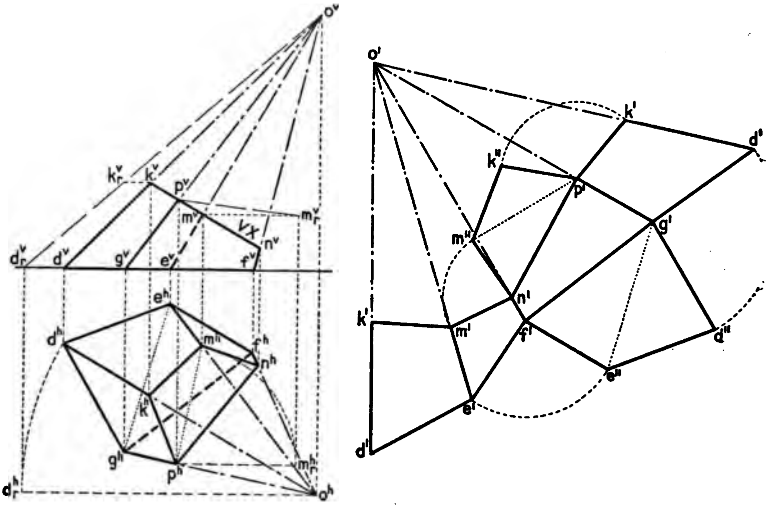

Fig. 234.

The constructions for finding the true lengths of the various lines are not shown in Fig. 234, except for the lines *od* and *mp*, here revolved parallel to *V*. Note that, when *od* is revolved, the point *k*, moving parallel to *H*, falls at k_r^v; hence $d_r^v k_r^v$ is the true length of the line *dk*.

Construction 7. To find the development of a cylinder or cone of revolution.

Example 1. Let it be required to develop the frustum of a cone of revolution placed in the third quadrant, Fig. 235. The cone is generated by the revolution, about the axis op, of the right line $o1$ making with the axis the angle δ. The distance traversed by point 1 is evidently equal to the circumference of the base of the cone. The development of the curved surface of the cone may be regarded as generated by a similar revolution of $o1$, but in the plane of the paper. With $o'1' = o^v1$ as radius, centre o', describe an arc, and upon it lay off the arc, $1'\ 5'\ 1'$, equal to the circumference of the base (A). The curved surface of the frustum is developed by means of the assumed elements as shown. Since the elements are all of the same length, the true length of any portion, as 2–10, of an element oblique to H and V may be obtained by revolving the element to coincide with a contour element, as shown at 2_r10_r.

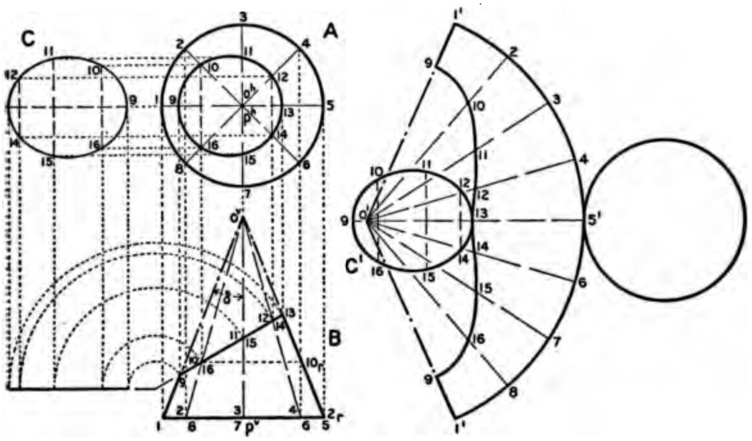

Fig. 235.

The measurements necessary for constructing, in the development, the true shape and size of the upper base, C', of the frustum, should be transferred from *A* and *B* with the dividers. The view *C*, obtained by revolving this base parallel to *H*, although unnecessary to the construction, is introduced to show more clearly the positions of the several measurements in *C′*.

Example 2. Let it be required to develop the frustum of a cylinder of revolution, placed in the third quadrant, Fig. 236.

The curved surface of the cylinder is developed by means of assumed elements, and each portion of the surface included between successive elements is usually treated as a plane surface; that is, the cylinder is regarded as a prism of an assumed number of sides, in the present case eight. A development thus obtained is not strictly accurate, although sufficiently so for most practical purposes. If greater accuracy is necessary, the length of the circumference of the base must be calculated, or found by rectifying a portion of the curve. The method of finding the development of the given cylinder will become apparent on scanning Fig. 236.

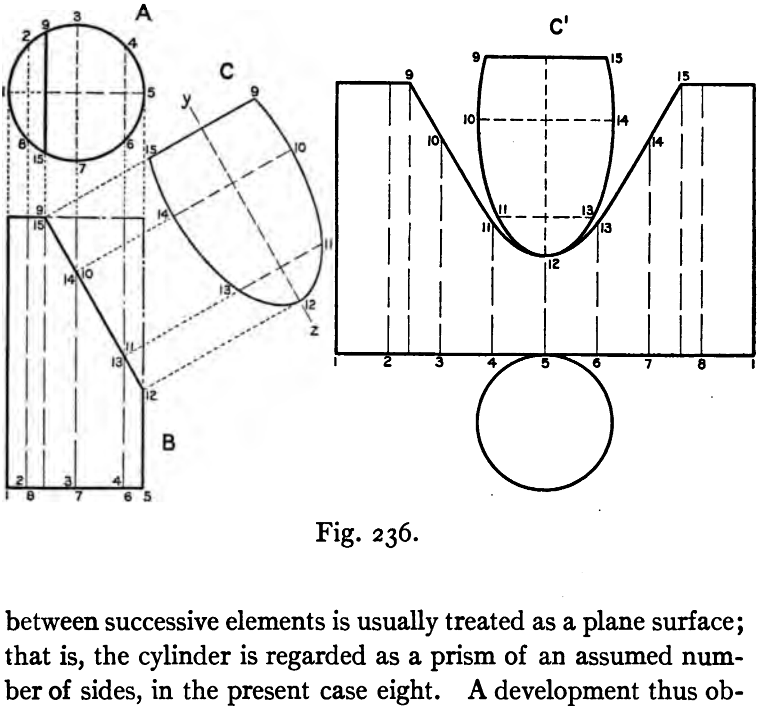

Fig. 236.

The measurements used in drawing the true section, *C′*, in the development, should be transferred from *A* and *B* with the dividers. The true section, *C*, although unnecessary, is introduced to indicate more clearly the positions in *A* and *B* of the necessary measurements in *C′*. It is obtained by revolving the section about *yz* until parallel to *V*, the widths 9–15, 10–14, and 11–13 being equal to 9–15, 3–7, and 4–6 in the plan. Note that the points 9 and 15 are located by means of additional elements.

Construction 8. Given a surface of revolution whose axis is perpendicular to H or V, to find its section when intersected by a plane perpendicular to H or V.

Since the given secant plane is perpendicular to *H* or *V*, one projection of the required section is a *straight line*, coinciding with the edge view of the secant plane, and is, therefore, a known quantity; that is, the construction is a special case of the following: Given one projection of a line lying in a surface of revolution, to find the other projection.

Assume auxiliary planes perpendicular to the axis of the given surface. Each auxiliary plane intersects the surface in a circle; and, since the axis of the surface is perpendicular to *H* or *V*, each circle projects as a straight line in one view, while the other view shows its true shape and size. By means of the intersections of the known projection of the given section and the corresponding projections of the auxiliary circles, the required projection can be found.

Example 1. A cone of revolution, Fig. 237, with its axis perpendicular to *H*, is intersected by a plane, *Q*, perpendicular to *V* and parallel to an element of the cone. Pass the auxiliary planes *R*, *S*, *T*, and *U*, perpendicular to the axis, *oz*, of the cone, and draw the plan of the circles cut from the cone. The known projection of the section, coincident with *VQ*, intersects the circles in the points 1, 2, 2′, . . . 5′. Projecting these points to the plan of the circles, the required plan of the section is obtained as shown.

The true section (a *parabola*) is obtained by revolving it parallel to *V* about an axis *xy* parallel to *VQ*. The widths 2—2′, . . . 5—5′, in the revolved section are evidently the same as 2—2′, . . . 5—5′ in the plan.

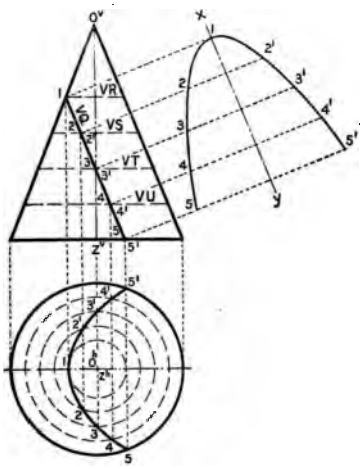

Fig. 237.

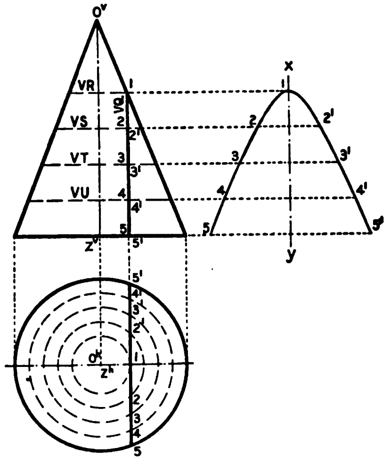

Fig. 238.

Example 2. The secant plane *Q*, Fig. 238, is parallel to the axis of the cone, and both projections of the section are straight lines. The plan and elevation of points in the outline of the section are found by means of the auxiliary circles, after which the true section, a *hyperbola*, can be shown by revolving it parallel to *V*.

The following examples illustrate applications of the same construction to surfaces of double curvature. Detailed explanation is omitted, as each construction should be evident from inspection of the figure.

The student should visualize the several steps and the resultant sections in each case.

Example 3. A sphere, placed in the first quadrant, Fig. 239, intersected by a plane perpendicular to *H* and inclined to *V*. The true section, a *circle*, is shown revolved parallel to *H*.

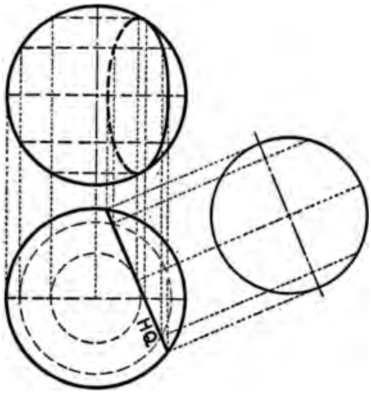

Fig. 239.

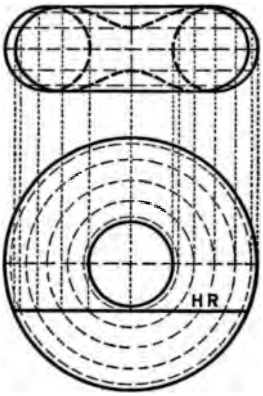

Fig. 240.

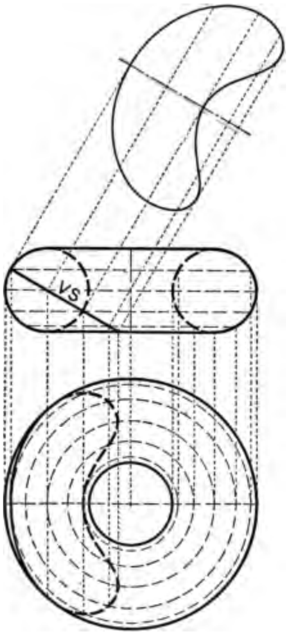

Fig. 241.

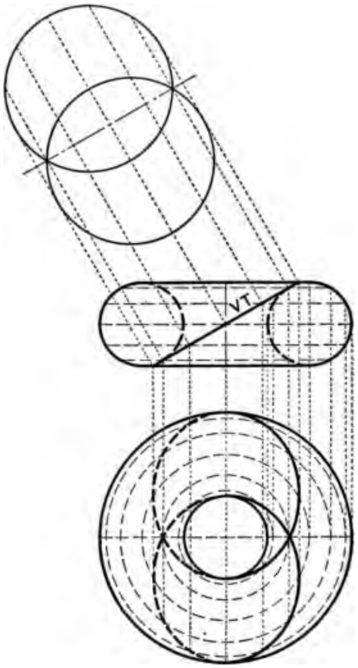

Fig. 242.

Example 4. A torus intersected by a plane perpendicular to *H* and parallel to *V*, Fig. 240. The top and bottom auxiliary planes are each tangent to the torus in a circle; each of the intermediate planes cuts two circles from the torus. The elevation shows the true shape of the section.

Example 5. A torus intersected by a plane perpendicular to *V* and inclined to *H*, Fig. 241. The true section is shown revolved parallel to *V*.

Example 6. *Special case.* A torus intersected by a plane perpendicular to *V* and doubly tangent to the surface, Fig. 242. In this case the true section consists of two equal intersecting circles.

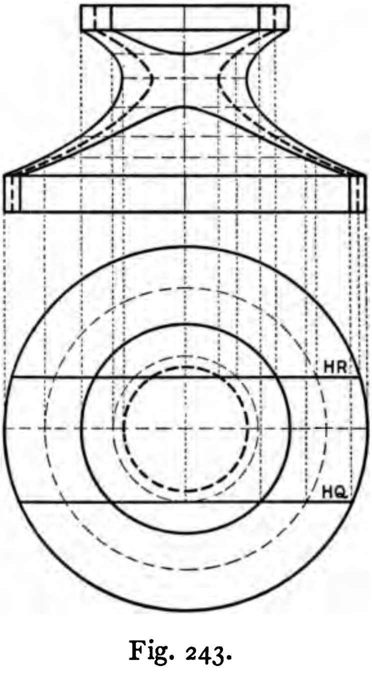

Fig. 243.

Example 7. A scotia, Fig. 243, intersected by two given planes, one of which is in front of the axis, and the other behind it. Both planes cut sections composed of two parts.

45. Intersection of Solids.

When one solid cuts into or penetrates another, the continuous line or lines formed by the intersection of their surfaces is called the intersection of the solids.

Construction 9. To find the intersection of two solids, when obtainable by means of auxiliary planes perpendicular to H or V.

This construction applies to all cases of solids bounded wholly by plane faces, and to some cases in which one or both of the solids are curved. The general case will be given in a later chapter.

If two solids intersect, planes can be passed through them which will cut intersecting sections. The points in which the boundaries of the sections intersect are common to both solids, and hence lie in the required intersection. When either solid is bounded by plane faces, it is usually most convenient to take the cutting planes through its edges.

EXAMPLE 1. Let it be required to find the intersection of a square pyramid and a triangular prism, Fig. 244. Since the solids are bounded by plane faces, the construction may be treated as an extension of Problem 10, *f*.

The construction should be begun by finding where *some* edge of either solid pierces the other solid. It may be assumed from inspection that the edge *K* of the prism intersects the pyramid. Through *K* pass the auxiliary plane *Q* perpendicular to *V*, and find the H-projection, 1–2–3–4, of the section cut from the pyramid. Line K^h intersects 1–4 at e^h, and 2–3 at f^h, which shows that *K* pierces the faces *aod* and *boc* of the pyramid.

Having found where *K* pierces the pyramid, the intersection of the face *KL* of the prism and the pyramid may next be considered. Through the edge *L*, pass the plane *R* perpendicular to *V*, and cutting the pyramid in the section 5–6–7–8. The line *L* intersects the edge 5–8 of the section at the point *g*, lying in the face *aod*, which contains also the point *e;* hence *eg* is the intersection of the faces *KL* and *aod*. The

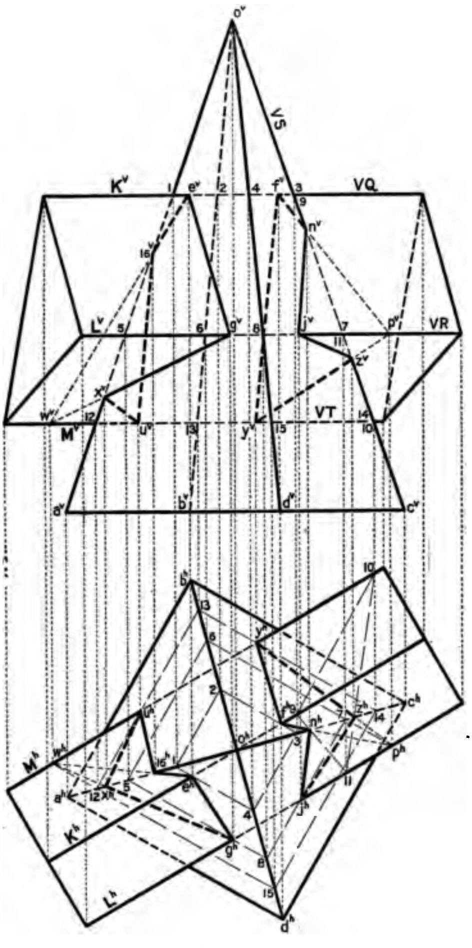

Fig. 244.

line *L* intersects the edge 7–8 of the section at point *j*, lying in the face *doc*. But *K* pierces the pyramid at *f*, lying in the face *boc*. It is evident then that *KL* intersects both of the right-hand faces of the pyramid, and that the line of intersection must cross the edge *oc;* that is, that *oc* must pierce the face

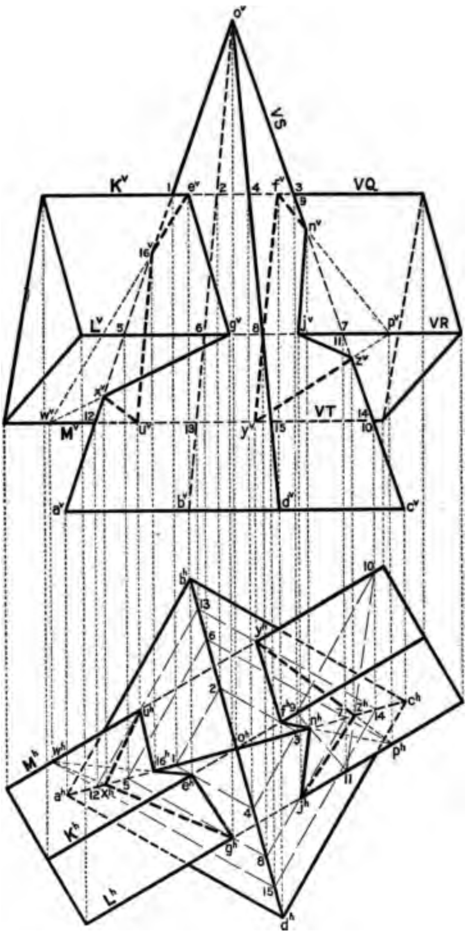

Fig. 244.

KL of the prism. Through *oc* pass the plane *S* perpendicular to *V*, cutting the prism in the section 9–10–11. Line o^hc^h intersects the edge 9–11 of the section in n^h, one projection of the point in which *oc* pierces *KL*. The line $f^h n^h j^h$ is therefore one projection of the intersection of *KL* and the faces *boc* and *doc*.

It is not imperative, however, to find the point *n* in this way, for the following construction may be used to advantage. If the edge 6–7 of the section lying in *R* be produced, its intersection, *p*, with *L*, will lie in the face *boc* produced. Connecting *p* with *f*, the point in which *K* pierces *boc*, gives the intersection of the planes of the faces *KL* and *boc*, which crosses the edge *oc* at point *n*. Similarly, if the edge 4–3 of the section in *Q* were produced, it would intersect K^h at a point (not shown in the figure) lying in *doc* produced; and, if this point were connected with *j*, this line, the intersection of the planes of *KL* and *doc*, would also pass through the point *n*.

The finding of the rest of the intersection of the solids is but a repetition of the foregoing processes, as follows: Pass plane *T*, perpendicular to *V*, through the edge *M*. It cuts from the pyramid the section 12–13–14–15. Line *M* intersects 12–13 at *u*, lying in the face *aob*, while *L* pierces *aod*. Hence the face *LM* intersects both *aob* and *aod*. Let it be decided to find the intersection of these faces by means of the plane of *aod*. Produce the H-projection of the edge 15–12 of the section in plane *T*. It cuts M^h at w^h, the H-projection of the point in which *M* pierces the plane of *aod*. The line *gw*, the intersection of *LM* and *aod* produced, cuts the edge *ao* at *x*, common to the faces *aod* and *aob;* hence *xu* is the intersection of *LM* and *aob*. The line *ew* is common to the face *KM* and *aod* produced. Hence the line *e*–16 is the intersection of *KM* and *aod*. The face *KM* cuts *ao* at 16, while *M* pierces *aob* at *u*. Hence 16–*u* is the intersection of *KM* and *aob*, which completes the intersection *egxu*–16 of the prism and the left-hand faces of the pyramid.

The edge *M* pierces *boc* at *y*, lying in the edge 13–14 of the section cut by plane *T*. The edge *K* also pierces *boc*, at *f*. Hence *fy* is the intersection of *KM* and *boc*. Edge *L* pierces *boc* produced at *p*, and *yz*, lying in *yp*, is the intersection of *LM* and *boc*. The point *z* is common to *doc* and *LM*, while

L pierces *doc* at *j*. Hence *zj*, the intersection of *LM* and *doc*, completes the intersection *jnfyz* of the solids.

Visibility. When the visibility of intersecting solids cannot be decided by inspection, it can be found according to Art. 38. On beginning the construction, it is of distinct advantage to determine which surfaces of the given solids would be visible if each solid were isolated, and then, as fast as points in the intersection are obtained, to indicate the visible lines by accenting them. It should be noted that, in either projection, the line of intersection of two faces cannot be visible unless both faces are visible in that projection. For example, in Fig. 244, the line $e^v g^v$ is visible, since it is the intersection of the faces *aod* and *KL*, both of which are visible in elevation. On strengthening $e^v g^v$, it becomes evident that the portion of $K^v L^v$ to the left of $e^v g^v$ is visible. Having lined in this portion of $K^v L^v$, it will be easy to see that the point 16^v, when found, must lie behind $K^v L^v$, and that 1–16^v and 16^v–e^v must be invisible.

EXAMPLE 2. The intersection of a prism and pyramid, placed in the first quadrant, Fig. 245. The auxiliary plane *Q*, perpendicular to *H*, taken through edges *A* and *C* of the prism, contains the edges *eo* and *ok* of the pyramid; hence points 1, 2, 3, and 4 are obtained by direct projection. The planes *R* and *S*, taken through *B* and *D* respectively, are perpendicular to *V*. The plan of the sections cut from the pyramid by these planes are intersected by B^h and D^h at 5, 6, 7, and 8 respectively. Projecting the points to the elevation, the intersection 1–7–4–5 may at once be completed.

It will be seen that, since *D* pierces the face *nom*, *AD* and *DC* intersect both *nom* and *mok*. Having found the point 2, common to *AD* and *mok*, a second point in the intersection of these two faces is obtained by means of the assumed plane *U* perpendicular to *V*, and cutting from the prism the auxiliary line *X*. The plan of the section cut from the pyramid by plane *U* is intersected by X^h at 9, the required second point. Similarly, having the point 3 common to *DC* and *mok*, a second point in the intersection of these faces is obtained by means of the assumed plane W, perpendicular to V, and so taken that the H-projection of the line Y, cut from DC, coincides with X. The line Y^h intersects the plan of the section

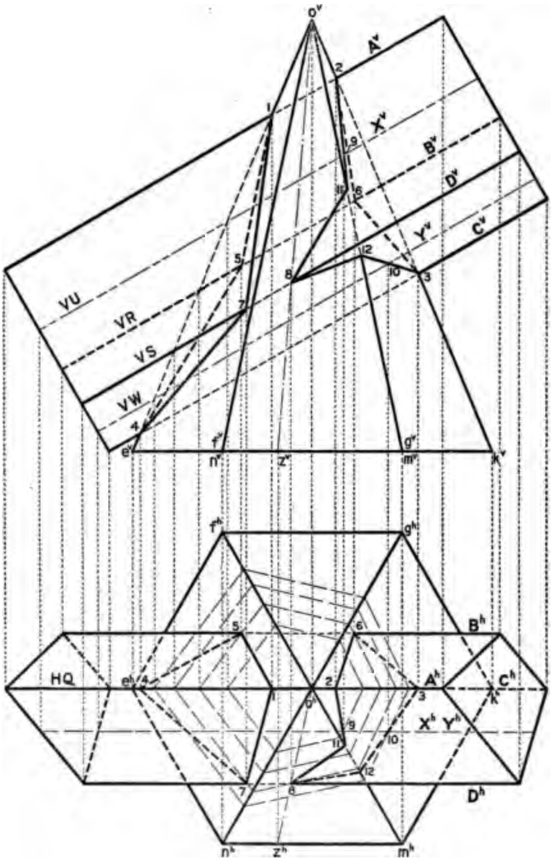

Fig. 245.

cut from the pyramid at point 10. Projecting 9 and 10 to the elevation, and drawing the lines 2–9–11 and 3–10–12, the intersection can be completed

Development of the pyramid. The construction (no figure) is similar to that of the pyramid given in Fig. 234. A point in the intersection not falling on an edge of the solid can usually be located to advantage by triangulation. Thus, for example, the point 8 might be located on the development by means of the true lengths of 8–11 and 8–12. When, however, the position of a point is such that accurate intersections in the triangulation cannot be readily obtained, then the point, as 8, may be located by means of an auxiliary line (o^hz^h, o^vz^v) passed through the point and the vertex of the pyramid.

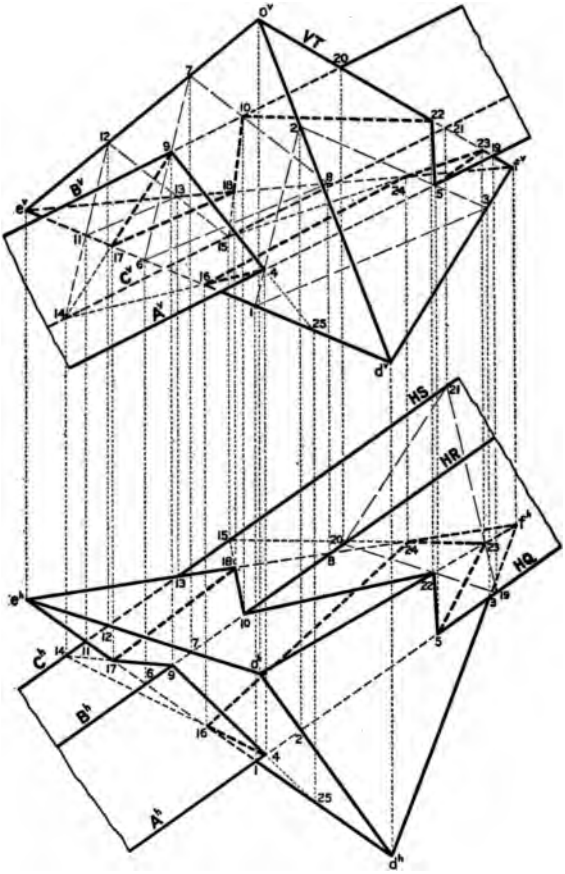

Fig. 246.

Exercise 20. (*a*) Draw the solids twice the size of Fig. 245. Find their intersection. (*b*) Develop each solid on cardboard or thick Manila paper, and make a model in which the prism shall accurately penetrate the pyramid.

Example 3. The intersection of a prism and pyramid, placed in the first quadrant, Fig. 246. The auxiliary planes

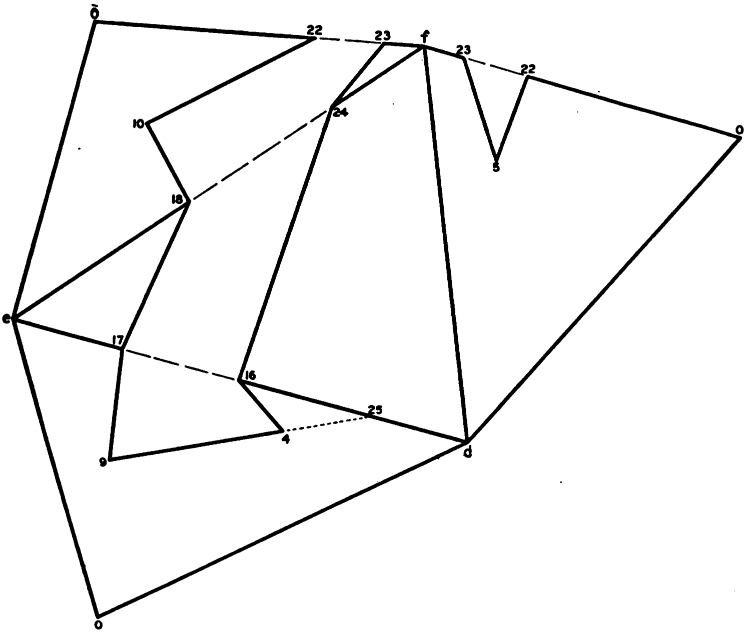

Fig. 247.

Q, R, and S, perpendicular to H, are taken through A, B, and C respectively. Plane Q cuts from the pyramid the section 1–2–3, which gives the elevation of points 4 and 5 of the intersection. Plane R cuts the section 6–7–8, giving the elevation of the points 9 and 10 of the intersection. Plane S cuts the section 11–12–13, which is not intersected by C. The lines 12–11 and 12–13 produced meet C at 14 and 15 respectively. Connecting points 4 and 9 with 14 gives the lines 4–16 and

9–17 of the intersection. Connecting 10 and 15 gives the line 10–18, after which 17–18 is drawn. Plane *T*, perpendicular to *V*, is passed through the edge *of* of the pyramid. This plane cuts the prism in the section 19–20–21, giving points 22 and 23 in the plan. Connecting 23 and 15 gives point 24, after which the intersection of the solids is completed by drawing 22–5, 23–5, 22–10, and 24–16.

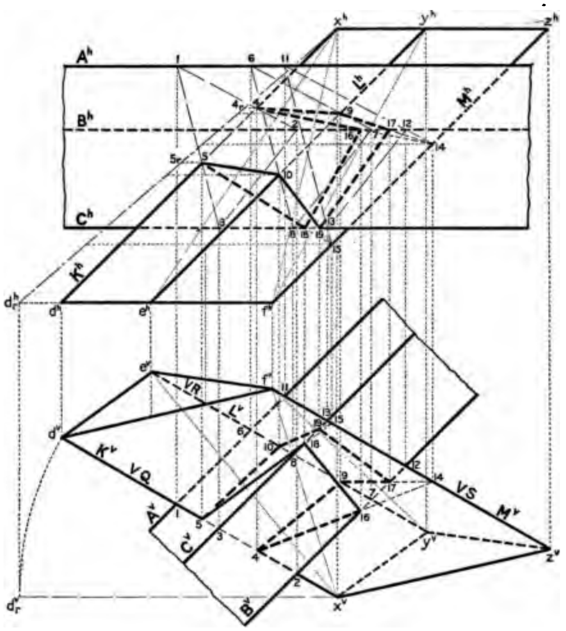

Fig. 248.

Development. The development of the pyramid is given in Fig. 247. The points in the intersection not falling on an edge of the solid are located by triangulation (*e.g.* point 4).

Exercise 21. (*a*) Draw the solids twice the size of Fig. 246. Find their intersection. (*b*) Make an accurate model of the intersecting solids.

EXAMPLE 4. The intersection of two triangular prisms placed in the third quadrant, Fig. 248. Auxiliary planes perpendicular to *V* are taken through the edges *K*, *L* and *M*, cutting the prism *ABC* in the sections shown. The points 4 on *K*, and 9 on *L*, are in the face *AB*, while 14 on *M* is in *AB* produced; drawing the triangle 4–9–14 gives points 16 and 17 on *B*. Likewise, the points 5 on *K*, and 10 on *L*, are in the face *AC*, while 15 on *M* is in *AC* produced; drawing the triangle 5–10–15 gives points 18 and 19 on *C*. Lines 16–18 and 17–19 complete the intersection.

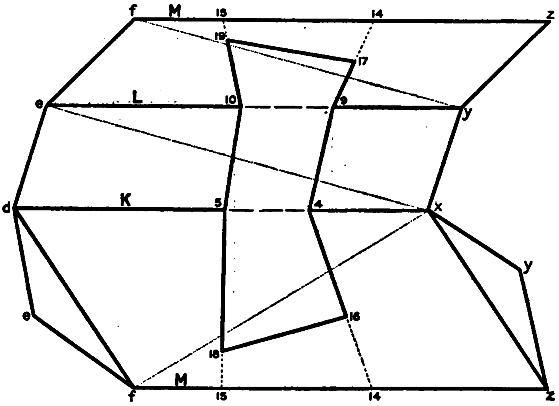

Fig. 249.

Visibility. The visibility of the faces of the prism *ABC* is determined by inspection. The visibility of the faces of *KLM* is determined by considering its front end, *def*, which shows that the edge *L* is above and behind the line *df* in the face *KM*.

Development. In Fig. 249 is given the development of the prism *KLM*. The lateral faces are developed by drawing a diagonal of each, as shown. The true length of *dx*, Fig. 248, revolved parallel to *H* at $d_r^h x^h$, serves also for *ey* and *fz*. Note that portions of these lines, as *d*5, *e*10, etc., may be measured by projecting them to $d_r^h x^h$ (compare with Fig. 234). The lines 5–18, 4–16, 10–19, and 9–17, in the development, Fig. 249, are located by means of the points 14 and 15, on line *M*, used in finding the intersection.

Method of assuming secant planes in the case of the sphere or torus. In the case of a sphere or torus, it is best to assume the auxiliary secant planes in pairs, each pair being placed equally distant from the centre of the solid, and hence cutting the solid in equal circles. In addition to such pairs of planes, it is usually necessary to assume single planes for the purpose of determining special points, as, for example, where the projection of an intersection is tangent to the boundary of one of the solids. Also, in assuming secant planes, it is often necessary to assume a circle lying in a desired position to repre-

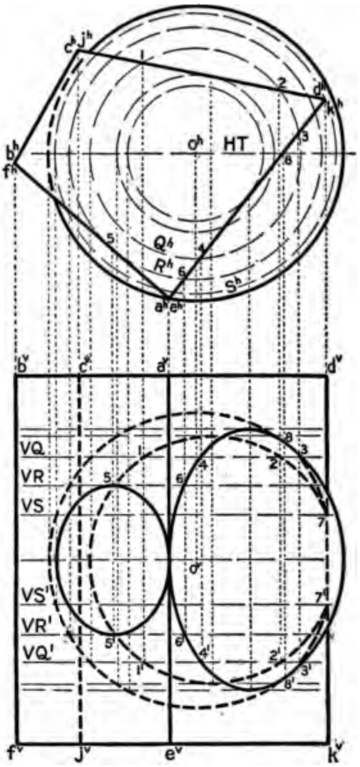

Fig. 250.

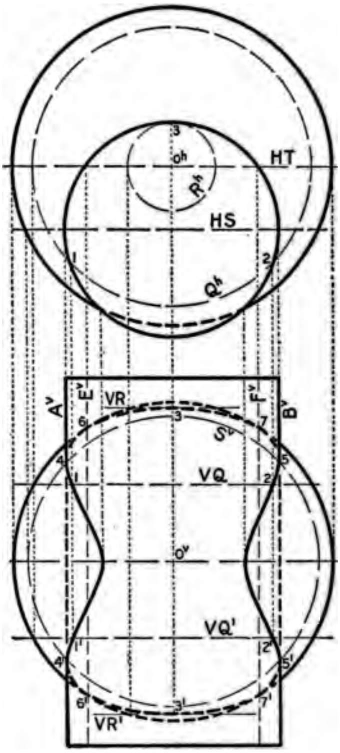

Fig. 251.

sent a section cut from the given solid, and then to find the position of the secant plane containing this circle.

EXAMPLE 5. The intersection of a sphere and an irregular right prism placed in the third quadrant, Fig. 250. Any auxiliary plane parallel to *H*, and intersecting both solids, cuts a circle from the sphere, and a quadrilateral from the prism. The intersection of the boundaries of these sections are points common to both solids and therefore lie in their intersection.

The pair of planes, *Q* and *Q′*, for example, cut from the sphere two equal circles, projected at Q^h, and from the prism a quadrilateral, the H-projection of which coincides with its plan, $a^hb^hc^hd^h$. By their intersection these sections give 1, 2, 3, 4, 1′, 2′, 3′, and 4′, points in the required intersection.

To find the highest and lowest points in the intersection of each face of the prism and the sphere: Take, for example, the face *abfe*. Draw, tangent to a^hb^h, the H-projection of the similar sections *R*, of the sphere. Find the planes, *VR* and *VR′*, which contain these sections, and project the point of tangency to 5 and 5′ in elevation. Note that the planes *R* and *R′*, when drawn, may be utilized to find points, as 6 and 6′, on the other faces of the prism.

The points 7 and 7′ on d^vk^v are found by means of the similar sections *S* and *S′*, obtained from S^h drawn through d^hk^h. The points 8 and 8′, where the intersection is tangent to the contour of the sphere, are found by using the auxiliary plane *T*, parallel to *V*, passed through the centre of the sphere.

EXAMPLE 6. The intersection of a cylinder and a sphere placed in the third quadrant, Fig. 251. Auxiliary planes parallel to *H*, as *VQ*, *VQ′*, cut circular sections from both solids. The intersection of these sections give points, as 1, 2, 1′, 2′, in the required intersection. The highest and lowest points, 3 and 3′, of the intersection lie in the planes *R* and *R′*, located from the assumed section R^h, tangent to the plan of the cylinder. The points 4, 5, 4′ and 5′, where the contour elements of the cylinder pierce the sphere, are obtained by means of the plane S, taken parallel to V, through the centre of the cylinder. The plane T, through the centre of the sphere and parallel to V, cuts the cylinder in the elements E^v and F^v, giving 6, 7, 6′ and 7′, points in which the intersection is tangent to the contour of the sphere.

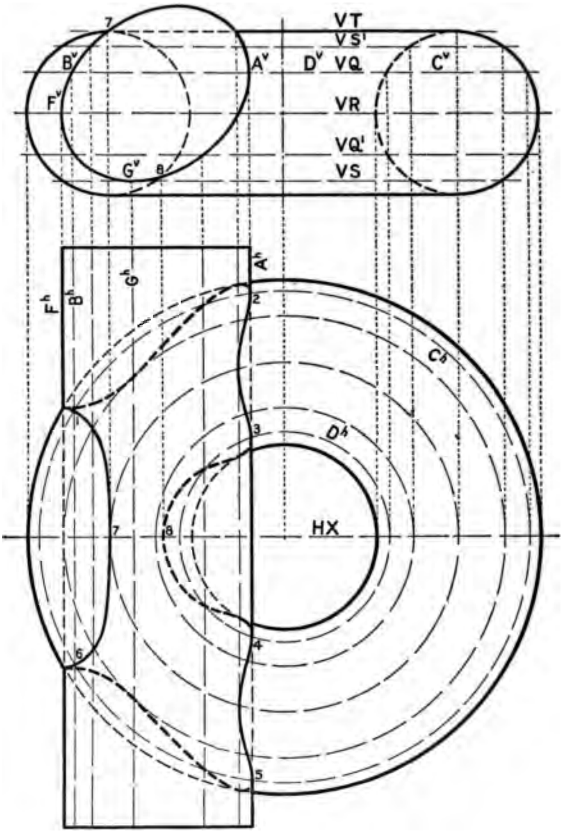

Fig. 252.

EXAMPLE 7. The intersection of a torus and an elliptical cylinder, placed in the first quadrant, Fig. 252. An auxiliary plane, as Q, parallel to H, cuts the cylinder in the elements A and B, and the torus in the circles C and D, giving six points, 1, 2, . . . 6, of the required intersection. The chosen secant planes, parallel to H, should include the following: one through each contour element, A and F, of the cylinder; one through the lowest element, G; one through the centre of the torus (in the present case this plane passes also through the contour element F); and one tangent to the top of the torus. The points 7 and 8 of the intersection are best obtained by means of the plane X, taken through the centre of the torus, and parallel to V.

EXAMPLE 8. The intersection of a cylinder and cone placed in the third quadrant, Fig. 253. Auxiliary planes taken perpendicular to H, and through the vertex of the cone, cut rectangles from the cylinder and triangles from the cone. Thus, for example, the plane Q cuts the sections, 1–2–3–4, from the cylinder, and 5–0–6, from the cone. The points 7, 8, 9, and 10, common to these sections, lie in the required intersection.

The following specially placed secant planes should be noted: The plane R, containing the contour element, A, of the cylinder; plane S, containing the contour element, C, of the cone; plane T, tangent to the cone, and cutting from it, instead of a triangle, the single element E; plane U, tangent to the cylinder, and cutting from it the single element F. Plane U here contains, also, the contour element D of the cone, and the point 13 of the intersection. Note that point 13 may also be obtained by direct projection, and that point 14 is so obtained.

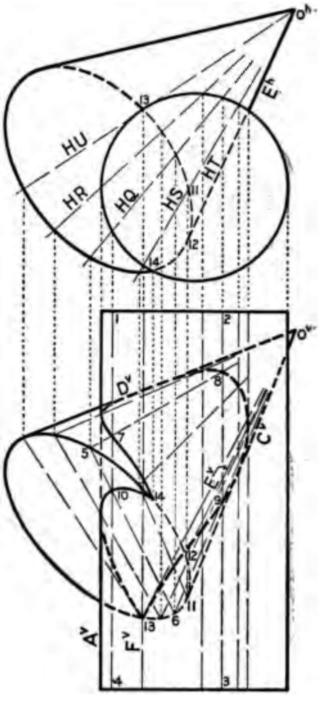

Fig. 253.

Example 9. The intersection of a cylinder and a rectangular prism placed in the third quadrant, Fig. 254. Planes parallel to *H* cut both solids in rectangular sections, the intersection of the boundaries of which are points in the required intersection. Since the cylinder is parallel to both *H* and *V*, the determination of the intersection of the cylinder and the auxiliary secant planes requires the use of a P-projection or side elevation.

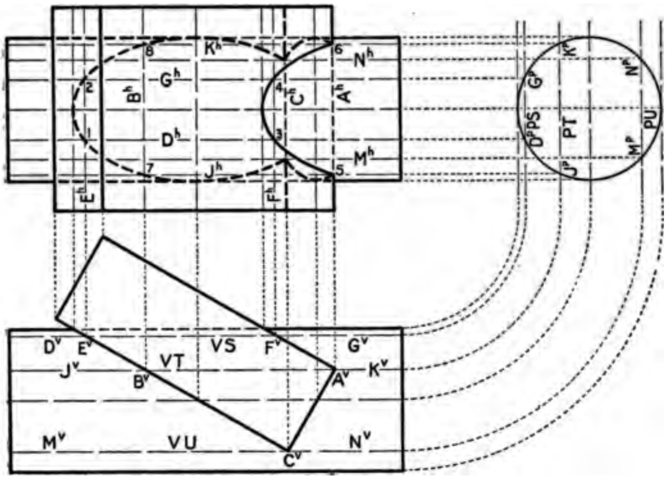

Fig. 254.

In assuming the secant planes, it is preferable to draw their P-projections first rather than the V-projections, except in the case of special planes, as *T* and *U*, taken through the edges *A* and *C* respectively, when the V-projections should be drawn first. The plane *S*, for example, cuts from the cylinder the elements *D* and *G*, and from the prism the lines *E* and *F*, giving by their respective intersections the points 1, 2, 3, and 4 of the required intersection. Drawing *VT* through A^v, *PT* is then found, the H-projections of the elements *J* and *K* are drawn, and the points 5 and 6 of the required intersection are found where these lines meet A^h.

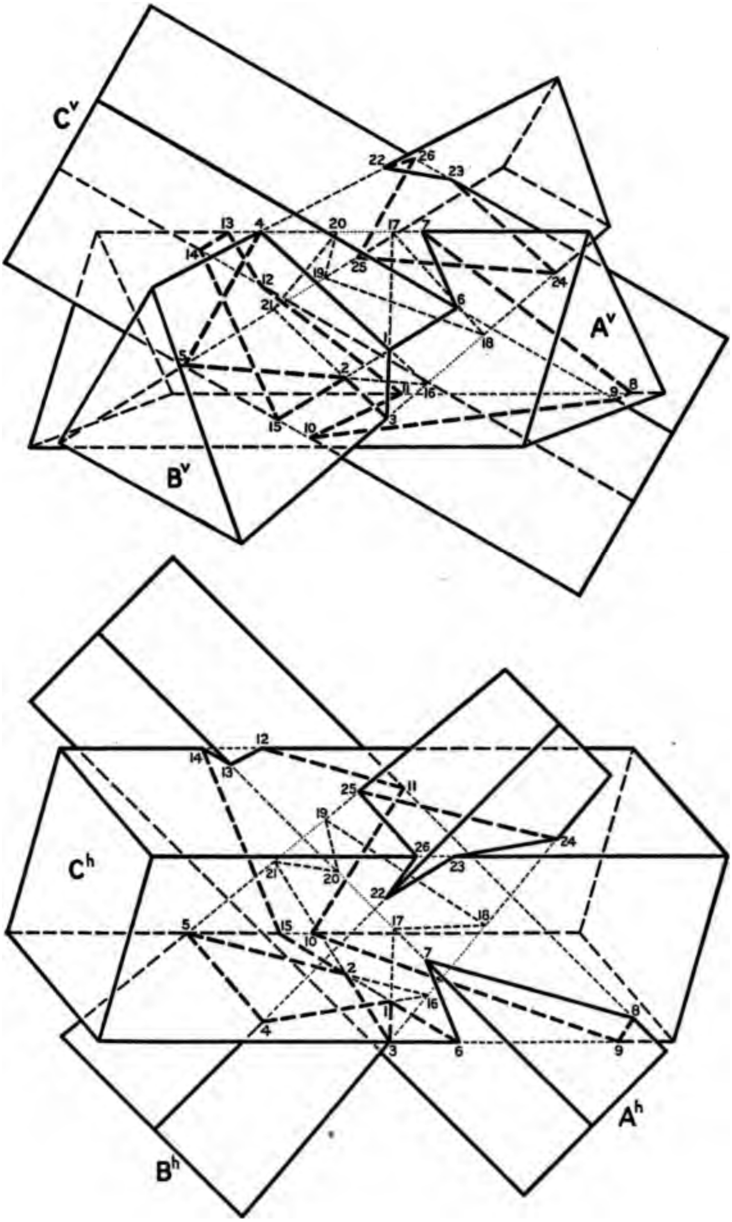
Cᵛ
Aᵛ
Bᵛ
Cʰ
Aʰ
Bʰ

Fig. 255.

EXAMPLE 10. The intersection of three solids, *A*, *B*, and *C*, placed in the first quadrant, Fig. 255. The solids intersect as follows:—

A, *B*, and *C* in the points 1 and 2.
A and *B* in 1–3–2.
B and *C* in 1–4–5–2.
A and *C* in 1–6–7–8–9–10–11–12–13–14–15–2.
B and *C* in 1–16–2 (intersection is inside of *A*).
A and *B* in 1–17–18–19–20–21–2 (inside of *C*).
B and *C* in 22–23–24–25–22.

This example is given to illustrate a complicated intersection, but explanation of the construction is omitted, as it is precisely the same as for two similar solids. The chance of confusion is greatly increased, naturally, by the greater number of construction lines attending the introduction of the third solid. In finding any complicated intersection, the work will become much easier if each step is inked in, with different colored inks, as the drawing proceeds.

APPENDIX.

REVIEW OF PRINCIPLES AND CONSTRUCTIONS INVOLVED IN THE PROBLEMS ON THE LINE AND PLANE.

Note.—The sign (*) placed after the statement of a principle signifies that the principle applies when the wording is changed, throughout, so that *horizontal* reads *vertical*, and *vice versa.*

The sign (†) placed before a construction indicates that the directions also apply when *horizontal* is changed to *vertical*, and *vice versa.*

General Principles.

If two lines in space are parallel, their projections on any plane are parallel.

If two lines in space intersect, their respective horizontal and vertical projections must intersect in points lying in the same ruled projector perpendicular to *GL*.

If one of two lines forming a right angle is parallel to a plane of projection, the right angle must project as such (unless the second line projects as a point).

If a line lies in a plane, the traces of the line lie in the traces of the plane; and conversely.

If the vertical projection of a line is parallel to *GL*, the line is parallel to *H;* and conversely.(*)

The lines in a plane which make the maximum inclination with *H* are perpendicular to the horizontal trace of the plane; and conversely.(*)

If a line and plane are perpendicular, the projections of the line are perpendicular to the traces of the plane; and conversely.

If two planes are parallel, their corresponding traces are parallel.

PROBLEM 1. *Given the projections of a line inclined to H and V, to find the true length of the line, and its true angles with H and V.*

First Method. (†) 1. Revolve the given line about an axis perpendicular to *H* until the line is parallel to *V*.

2. The new vertical projection is equal to the true length of the line, and makes with GL the angle which the line in space makes with H.

Second Method. (†) 1. Pass through the given line a plane perpendicular to H.

2. Revolve this plane about its horizontal trace into H.

3. The revolved position of the given line is equal to its true length, and makes with the trace of the plane the angle which the line in space makes with H.

PROBLEM 2. *Given the true angle which a line makes with one co-ordinate plane, and the angle which the projection on that plane makes with GL, to draw the projections of the line.*

(†) Let the angle with H, and the angle which the horizontal projection makes with GL, be given.

1. Place the line parallel to V, and making the given angle with H.

2. Revolve the line about an axis perpendicular to H until the horizontal projection takes the required position.

PROBLEM 3. *To draw the projections of a line making given angles with H and V.*

1. Locate the fixed end of the line.

2. Draw the projections of the line placed parallel to H, and making the given angle with V.

3. Revolve the line about an axis passing through the fixed end and perpendicular to V.

4. Draw the projections of the line placed parallel to V, and making the given angle with H.

5. Revolve the line about an axis passing through the fixed end and perpendicular to H.

6. Find the point of intersection of the two paths of revolution.

7. Connect this point with the fixed end of the line.

PROBLEM 4. *Given the projections, to find the traces of a straight line.*

Principle. If a point lies in a line, the projections of the point lie in the projection of the line.

Principle. If the vertical projection of a point lies in *GL* the point lies in *H*. (*)

PROBLEM 5. *Given the traces, to find the projections of a line.*

Principle. If a point lies in *H*, its vertical projection lies in *GL*.(*)

PROBLEM 6. *To pass a plane through two intersecting or two parallel lines.*

General Case. 1. Find the traces of each line.

2. Connect the corresponding traces.

Special Case. Principle. If a line is parallel to *H*, the horizontal trace of any plane containing the line will be parallel to the horizontal projection of the line.(*)

PROBLEM 7. *Given one projection of a point or a line lying in a plane, to find the other projection.*

General Case. Principle. If a line lies in a plane, the traces of the line lie in the traces of the plane.

Special Case. Principle. If the horizontal projection of a line lying in a plane is parallel to the horizontal trace of the plane, the vertical projection of the line is parallel to *GL;* and conversely.(*)

PROBLEM 8. *To find the position of a point or a line lying in a plane, when the plane is revolved into H or V.*

Principle. If a plane is revolved about its horizontal trace into *H*, any point lying in the trace remains fixed, and any point not in the trace moves in a plane perpendicular to the trace.(*)

Principle. The true distance from a point in a plane to the horizontal trace of the plane is the hypothenuse of a right triangle, whose base is the distance from the trace to the horizontal projection of the point, and whose altitude is the distance from *GL* to the vertical projection of the point.(*)

PROBLEM 9. *To find the intersection of two planes.*

General Case. Principle. If two planes intersect, the traces of the line of intersection lie at the intersection of the corresponding traces of the planes.

Special Case. Principle. If two intersecting planes are cut by a third plane, the lines of intersection meet in the intersection of the given planes.

PROBLEM 10. *To find the intersection of a line and a plane.*

General Method. 1. Pass any auxiliary plane through the given line.

2. Find the intersection of the two planes.

3. This intersection cuts the given line in the required point.

Usual Method. Take the auxiliary plane perpendicular to *H* or *V*.

PROBLEM 11. *To find the projections and true length of the shortest distance from a point to a plane.*

1. From the given point drop a perpendicular to the given plane.

2. Find the point where the perpendicular meets the given plane.

3. Find the true length of the perpendicular between the point in the plane and the given point.

PROBLEM 12. *To project a line upon an inclined plane.*

1. From each end of the given line drop perpendiculars to the given plane.

2. Find the projections of the points in which the perpendiculars meet the given plane.

3. Connect the respective projections of the points.

Problem 13. *Through a given point, to pass a plane which shall be parallel to a given plane.*

1. Through the given point draw an H- or V-parallel, parallel to the given plane.

2. Pass the required plane through the parallel.

Problem 14. *Through a given point, to pass a plane which shall be perpendicular to a given line.*

1. Through the given point draw an H- or V-parallel of the required plane.

2. Pass the required plane through the parallel.

Problem 15. *Through a given line, to pass a plane which shall be parallel to another given line.*

1. Through any point of the first line draw a line parallel to the second line.

2. Pass the required plane through the intersecting lines.

Problem 16. *Through a given point, to pass a plane which shall be parallel to each of two given lines.*

1. Through the given point draw lines respectively parallel to the given lines.

2. Pass the required plane through the intersecting lines.

Problem 17. *Through a given line, to pass a plane which shall be perpendicular to a given plane.*

1. Through any point of the given line draw a line perpendicular to the given plane.

2. Pass the required plane through the intersecting lines.

Problem 18. *To find the angle between two intersecting lines and the projections of the bisector of the angle.*

(†) 1. Find the horizontal trace of the plane containing the given lines.

2. About this trace, revolve the given lines into H.

3. Bisect the angle formed by the given lines revolved.

4. Counter-revolve the bisector.

Principle. In counter-revolving a plane about its trace, any known point in the revolved plane returns to its original position, and any point in the trace of the plane remains fixed.

PROBLEM 19. *To find the angle between a line and an inclined plane.*

1. From any point in the given line drop a perpendicular to the given plane.

2. Find the true angle between the perpendicular and the given line.

3. The required angle is the *complement* of the preceding angle.

PROBLEM 20. *Given one projection and three points in the other projection of a plane figure, to find the true shape and size, and to find the projections of the bisector of one angle of the figure.*

1. Find the plane of the figure by passing a plane through the three points whose horizontal and vertical projections are given.

2. Find the omitted projections of the remaining points or lines of the figure.

3. Revolve the resultant figure about either trace of the plane into a co-ordinate plane.

4. Bisect the given angle in the revolved figure.

5. Counter-revolve the bisector.

PROBLEM 21. *To find the projections and true length of the shortest distance from a given point to a given line.*

First Method. (†) 1. Find the horizontal trace of the plane containing the given line and point.

2. About this trace revolve the line and point into *H*.

3. Find the required true length by dropping a perpendicular from the revolved point to the resolved line.

4. To find the required projections, counter-revolve the perpendicular.

Second Method. 1. Assume an auxiliary *V* plane, *parallel* to the given line.

2. Project the given line and point on the auxiliary *V*.

3. On the auxiliary *V* draw the projection of the perpendicular from the given point to the given line.

4. Find the projections of the perpendicular.

5. Find the true length of the perpendicular.

PROBLEM 22. *To find the projections and true length of the shortest distance between two lines not in the same plane.*

Call either line *A* and the other line *B*.

1. Pass a plane through *A* parallel to *B*.

2. From any point in *B* drop a perpendicular to the plane.

3. Move the perpendicular parallel to itself until its foot rests on *A*.

4. Find the true length of the perpendicular.

PROBLEM 23. *To find the angle between a given plane and either co-ordinate plane.*

(†) To find the angle with *H*:—

1. Pass an auxiliary plane perpendicular to the horizontal trace of the given plane.

2. Find the true size of the right triangle lying in the auxiliary plane, and included between *H*, *V*, and the given plane.

3. The required angle is the angle which the hypothenuse of the triangle makes with *H*.

PROBLEM 24. *Given one trace of a plane, and the angle which the plane makes with either co-ordinate plane, to find the other trace of the plane.* (*Two cases.*)

Case I. (†) Let the horizontal trace of the plane and the angle with *H* be given.

1. Pass an auxiliary plane perpendicular to the given trace.

2. Using the given angle, construct the revolved position of a right triangle lying in the auxiliary plane, and included between *H*, *V*, and the required plane.

3. Find the vertical trace of the hypothenuse of the triangle (two results).

4. Draw the required vertical trace of the plane (two results).

Case II. (†) Let the vertical trace of the plane and the angle with *H* be given.

1. Draw the vertical trace of an auxiliary plane perpendicular to *H* and to the required plane.

2. With the vertical trace of the auxiliary plane as altitude and using the given angle, draw in *V* the revolved position of a triangle lying in the auxiliary plane and included between *H*, *V*, and the required plane.

3. Revolving the triangle about its vertical edge, draw in *H* the circle described by the lower end of the hypothenuse.

4. Tangent to this circle, draw the horizontal trace of the required plane (two results).

PROBLEM 25. *Given the angles which a plane makes with H and V, to find the traces of the plane.*

1. With any point, *o*, on *GL* as centre, and any radius, draw a circle. Through *o* draw a line perpendicular to *GL*.

2. Tangent to the circle, place the plane perpendicular to *V* and making the given angle with *H*.

3. Revolving the plane about an axis perpendicular to *H*, through *o*, note the fixed point through which the V-trace must pass, and draw the arc to which the H-trace must be tangent.

4. Tangent to the circle, place the plane perpendicular to *H* and making the given angle with *V*.

5. Revolving the plane about an axis perpendicular to *V*, through *o*, note the fixed point through which the H-trace must pass, and draw the arc to which the V-trace must be tangent.

hrough the proper fixed points, and tangent to the proper aw the traces of the required plane.

LEM 26. *To find the true size of the angle between two lanes.*

ind the horizontal projection of the line of intersection given planes.

2. Take an auxiliary *V* through the line of intersection.

3. Find the projection of the line of intersection of the given planes on the auxiliary *V*.

4. Draw the horizontal and auxiliary-vertical traces of a plane perpendicular to the intersection of the given planes.

5. Find the required angle by revolving into *H* the lines cut from the given planes by the perpendicular plane.

PROBLEM 27. *Given a plane, and a point in the plane revolved into H or V, to find the projections of the point.*

(†) Let the given point lie in *H*.

1. Through the given point pass an auxiliary plane perpendicular to the horizontal trace of the given plane.

2. Revolve into *H* the line of intersection of the auxiliary and given planes, and produce the line in both directions.

3. Revolve the given point into the revolved line of intersection (two results).

4. Find the required horizontal projections of the point by counter-revolving the line of intersection.

5. Find the required vertical projections of the point by means of H-parallels.

l.c.